A Transition of Complex Hyperbolic Space

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Abstract

By degenerating algebraic structure of $\mathbb{C}$ we construct a transition of geometries from complex hyperbolic space to a new geometry built from $\mathbb{R}\mathsf{P}^n$ and its dual. This transition provides a geometric context for considering the flexing of hyperbolic orbifolds, as defined by Cooper Long and Thistlethwaite. As an application, we connect the convex projective and complex hyperbolic deformations of triangle groups via this transition.

1 Introduction

In this paper we use the techniques of transitional geometry to shed light on an interesting phenomenon in the deformation theory of hyperbolic manifolds. Specifically, we will construct a one parameter family of geometries which deforms the geometry of complex hyperbolic space into a new homogeneous geometry described in Section 2, dubbed *self dual real projective space*.

Theorem 1.1. *For each $n > 0$ there is a one parameter family of homogeneous geometries $\mathbb{H}^n_\delta$ of dimension $2n$ depending on $\delta \in \mathbb{R}$. The isomorphism type of $\mathbb{H}^n_\delta$ depends only on the sign of δ: being isomorphic to complex hyperbolic space for $\delta < 0$, a fibered geometry over $\mathbb{H}^n_\mathbb{R}$ for $\delta = 0$, and self dual real projective space for $\delta > 0$.*

For each δ, there is an embedding $\mathbb{H}^n_\mathbb{R} \hookrightarrow \mathbb{H}^n_\delta$ as a totally geodesic half-dimensional submanifold, so this transition may be thought of as a deformation, degeneration and transition of complex hyperbolic space occurring around an invariant subgeometry isomorphic to $\mathbb{H}^n_\mathbb{R}$. To formally construct this transition, we develop a notion of *families* in geometric topology inspired by similar theories in algebraic geometry.

Definition 1.2. A family of spaces is a smooth manifold $\mathcal{X}$ equipped with a submersion $\pi\colon \mathcal{X} \to \mathbb{R}$. A *family of groups* is a family of spaces $\pi\colon \mathcal{G} \to \mathbb{R}$ together with a smooth section $e\colon \mathbb{R} \to \mathcal{G}$ and smooth maps $\mu\colon \mathcal{G} \times_\mathbb{R} \mathcal{G} \to \mathcal{G}$, $\iota\colon \mathcal{G} \to \mathcal{G}$ such that for each δ, the space $\mathcal{G}_\delta$ is a group with identity $e(\delta)$, multiplication $\mu|_{\mathcal{G}_\delta \times \mathcal{G}_\delta}$ and inversion $\iota|_{\mathcal{G}_\delta}$.

A *family of geometries* is given by a family of groups acting fiberwise transitively on a family of spaces, encoded as a family of groups together with a family of closed stabilizer subgroups for convenience. We construct the transition of Theorem 1.1 as follows. Recall the standard construction of complex hyperbolic space as the action of $\mathsf{SU}(n,1)$ on the projectivized hyperboloid for the $(n,1)$ Hermitian form on $\mathbb{C}^{n+1}$. We then construct a family of algebras transitioning from the algebraic structure of $\mathbb{C}$ to that of $\mathbb{R} \oplus \mathbb{R}$, and mimic the definition of $\mathbb{H}^n_\mathbb{C}$ along the way. The theme introduced by this example, using families of algebras to construct new families of geometries applies quite widely; a vast generalization of this will be presented in a forthcoming paper.

As the endpoint of this transition, self dual real projective space, can alternatively be constructed using $\mathbb{R}\mathsf{P}^n$ and its dual, this transition provides a connection between the seemingly disparate theories of complex hyperbolic and convex real projective deformations of real hyperbolic manifolds. This transition shows these two deformation theories of hyperbolic manifolds are in some sense deformations of each other.

As an example we consider hyperbolic triangle groups, which are known to be rigid in $\mathbb{H}^2_\mathbb{R}$ but deform nontrivially in both complex hyperbolic and real projective space. This provides a geometric way to understand the work of Cooper, Long and Thistlethwaite in [CLT07]

1.1 Deforming Hyperbolic Manifolds

The Teichmuüller theory of hyperbolic surfaces has enjoyed great success, and serves as a motivating case study for the deformation theory of more general geometric structures on manifolds. Given the importance of hyperbolic 3-manifolds in geometric topology following the work of Thurston, one may hope for an analogous theory in dimension 3 and higher. But as Mostow-Prasad rigidity implies he deformation space of finite-volume hyperbolic structures on an n-manifold is either empty or a single point for n at least 3, the naïve generalization of Teichmüller space carries no information. Instead, fruitful extensions arise from considering deformations into geometries *containing* hyperbolic space.

To discuss this properly, some terminology is in order. A homogeneous geometry $\mathbb{X} = (G, K)$[1] is a *subgeometry* of $\mathbb{Y} = (H, C)$ if $G \leqslant H$ and $K \leqslant C$. Dually, we call $\mathbb{Y}$ an *enveloping geometry* of $\mathbb{X}$. For each enveloping geometry $\mathbb{X} = (G, K)$ of hyperbolic space we can consider the deformation theory of hyperbolic structures, thought of as $\mathbb{X}$ structures via the containment $\mathsf{Isom}(\mathbb{H}^n) \leqslant G$. A hyperbolic structure $\rho\colon \pi_1(M) \to \mathsf{Isom}(\mathbb{H}^n)$ admits nontrivial $\mathbb{X}$ deformations if there are nearby representations $\rho'\colon \pi_1(M) \to G$ which are not conjugate into $\mathrm{Hom}(\pi_1(M), \mathsf{Isom}(\mathbb{H}^n))$.

Bending Deformations

Hyperbolic n space embeds $\mathbb{H}^n \hookrightarrow \mathbb{H}^{n+1}$ as a totally geodesic hypersurface, providing a first example of a deformation theory. Considering the case $n = 2$ corresponds to deformations of hyperbolic surface groups in $\mathsf{Isom}(\mathbb{H}^2) \cong \mathsf{PSL}(2, \mathbb{R})$ into $\mathsf{Isom}(\mathbb{H}^3) \cong \mathsf{PSL}(2, \mathbb{C})$ which is the classical theory of *quasifuchsian deformations*. Even in the case of already-flexible hyperbolic surfaces, extending the deformation problem to $\mathsf{Isom}(\mathbb{H}^3)$ yields many new deformations. Indeed the simultaneous Uniformization theorem of Bers [Ber60] identifies the space of marked quasifuchsian groups with the product of two copies of Teichmuller space $\mathcal{T}(S) \times \mathcal{T}(S)$, with the embedding $\mathbb{H}^2 \hookrightarrow \mathbb{H}^3$ inducing the diagonal embedding $\Delta\colon \mathcal{T}(S) \to \mathcal{T}(S) \times \mathcal{T}(S)$. In higher dimensions, analogous *bending* deformations can be constructed of $\pi_1(M^n) \hookrightarrow \mathsf{Isom}(\mathbb{H}^{n+1})$ along a totally geodesic hypersurface $\Sigma^{n-1} \subset M^n$.

Complex Hyperbolic Deformations

Complex hyperbolic space $\mathbb{H}^n_\mathbb{C}$ is defined analogously to the hyperboloid model of real hyperbolic space by the projective action of $\mathsf{SU}(n, 1)$ on an open ball in $\mathbb{C}\mathsf{P}^n$. The resulting $2n$ dimensional Riemannian manifold has constant holomorphic sectional curvature -1, with real sectional curvature quarter pinched $\in [-1, -1/4]$. The inclusion $\mathbb{R} \hookrightarrow \mathbb{C}$ induces inclusions $\mathbb{R}\mathsf{P}^n \hookrightarrow \mathbb{C}\mathsf{P}^n$ and $\mathsf{SO}(n, 1) \leqslant \mathsf{SU}(n, 1)$, realizing complex hyperbolic space as an enveloping geometry for real hyperbolic n space (henceforth $\mathbb{H}^n_\mathbb{R}$ to avoid confusion) of codimension n. The corresponding *complex hyperbolic deformations* have been extensively studied and provide a natural source of potential deformations in any dimension. For a sample of work concerning complex hyperbolic surface groups, see [RPP10, CRPS16], and [Sch01, R.E, RPWX16] for triangle groups.

[1] We encode a homogeneous space here by its automorphism group and a chosen point stabilizer. The space itself is reconstructed as $X = G/K$.

Convex Real Projective Deformations

The identification $\mathsf{Isom}(\mathbb{H}^n) \cong \mathsf{PSO}(n,1)$ provides another avenue for deformation coming from the natural inclusion $\mathsf{SO}(n,1) \leqslant \mathsf{SL}(n+1;\mathbb{R})$. The corresponding deformation theory considers *convex real projective manifolds* or quotients of open convex domains in $\mathbb{R}\mathsf{P}^n$ by properly discontinuous group actions. The study of these convex real projective structures has recently been of much interest, evidenced by the work of Benoist [Ben06], Ballas & Danciger [Bal, BDL15] and others.

One main distinction between this and the previous two cases is the enveloping geometry does not contain $\mathbb{H}^n_{\mathbb{R}}$ as a *lower dimensional subset.* In some sense this is the closest to the original spirit of Teichmüller theory as the manifolds considered equidimensional with the original hyperbolic manifold M^n, and not of higher dimension as in the bending and complex hyperbolic examples. In the case of 2-orbifolds, Goldman and Choi computed the dimension of the deformation space into $\mathsf{SL}(3,\mathbb{R})$; recent work of Long and Thistlethwaite extends this to $\mathsf{SL}(n;\mathbb{R})$ for triangle groups. The situation in dimension three and above remains more mysterious. In [CLT06], numerical calculations showed that only slightly over 1% of the first 4500 closed manifolds in the Hogdson-Weeks census flex nontrivially as real projective structures.

1.2 Flexing Closed Hyperbolic Manifolds

The complex hyperbolic and convex real projective deformation theories have developed more or less in parallel, and a passing glance reveals no obvious similarity between them. Indeed, the manifolds considered in the complex hyperbolic deformation theory are of twice the dimension of their convex real projective counterparts. However, a mysterious pattern emerges when considering the question of *which* manifolds admit $\mathbb{R}\mathsf{P}^n$ or $\mathbb{H}^n_{\mathbb{C}}$ structures, respectively. Explicit computation of representation varieties for hyperbolic 3-manifolds $\mathrm{Hom}(\pi_1(M),\mathsf{SL}(4,\mathbb{R}))$ by Cooper, Long and Thistlethwaite [CLT06] led to further work [CLT07] containing the following theorem.

Theorem 1.3 (Cooper-Long-Thistlethwaite)**.** *Suppose that $\rho\colon \Gamma \to \mathsf{SO}_0(n,1)$ is a representation of a closed hyperbolic manifold which is a smooth point of the representation variety $V_{\mathsf{SL}} = \mathrm{Hom}(\Gamma,\mathsf{SL}(n{+}1;\mathbb{R}))$. Then ρ is a smooth point of $V_{\mathsf{SU}} = \mathrm{Hom}(\Gamma,\mathsf{SU}(n,1))$ and further $\dim_{\mathbb{R}}(V_{\mathsf{SL}}) = \dim_{\mathbb{R}}(V_{\mathsf{SU}})$ near ρ, and conversely.*

Following the analysis in [CLT07], this has the surprising consequence that a closed hyperbolic manifold with holonomy ρ admits real projective deformations if and only if the complex hyperbolic manifold with the same holonomy deforms nontrivially. Precisely *which* hyperbolic manifolds admit convex real projective or complex hyperbolic deformations remains mysterious, but this result provides a strong tie between the two theories.

This paper proposes that the correct light in which to view this surprising connection is through a particular geometric transition. That in fact, these two deformation theories for hyperbolic manifolds are themselves deformations of each other. To make this precise, we pull inspiration from the field of transitional geometry to construction a *geometric transition of enveloping geometries*, corresponding to a *transition of enveloping groups* beginning with the pair $\mathsf{SO}(n,1) \leqslant \mathsf{SU}(n,1)$ and continuously deforming into the pair $\mathsf{SO}(n,1) \leqslant \mathsf{SL}(n+1;\mathbb{R})$.

1.3 Transitional Geometry

A geometric transition is a continuous path (G_t, X_t) of geometries where the isomorphism type is discontinuous in t. The example that inspires the theory is the continuous family of simply connected model spaces $\mathbb{M}_\kappa$ of constant curvature κ, which are isomorphic to the hyperbolic space for $\kappa < 0$ and the sphere for $\kappa > 0$, transitioning through Euclidean space at $\kappa = 0$, which was already known to Klein. More recently, the work of Danciger discovered a new transition connecting hyperbolic space to its Lorentzian, analog Anti deSitter space AdS^n, [Dan11]. Inspired by this, Cooper, Danciger and Wienhard developed the theory of *conjugacy limits* formalizing a means of constructing geometric transitions, subject to the constraint that all geometries in question arise as subgeometries of some fixed ambient space [CDW14]. In particular, applying this to subgeometries of real projective space they completely classified the isomorphism types of conjugacy limits for the indefinite orthogonal groups $\mathsf{O}(p,q) \leqslant \mathsf{GL}(p+q;\mathbb{R})$. This determines the possible degenerations of $\mathbb{H}^n = (\mathsf{SO}(n,1), \mathbb{B}^n)$ in $\mathbb{R}\mathsf{P}^n$, recovering both the classical transition to spherical geometry through Euclidean, and the aforementioned work of Danciger.

One motivation for studying these objects is that transitional geometries provide a means to "save" certain geometric structures from collapse. A path of (G,X) structures on M is said to *collapse* in $\mathcal{D}_{(G,X)}(M)$ if the charts limit to submersions onto a lower dimensional submanifold $Y \subset X$ and the transition maps converge to elements of G preserving Y. Often such collapse signals the possibility of a transition, achieved through rescaling the degenerating (G,X) structure to converge to a limiting geometry. Hodgson [Hod86] and Porti [Por98] analyze Euclidean limits resulting from hyperbolic conemanifolds collapsing to a point, which plays an important role in the Orbifold Theorem of Cooper, Hodgson, & Kerckhoff [CHK00] and Boileau, Leeb & Porti [PLB05]. Further work of Porti studies the nonuniform collapse of hyperbolic structures to Nil [Por03] and Sol [HPS01], and in a series of papers Danciger considers the collapse of hyperbolic structures onto a codimension one hyperbolic spaces and the associated Anti de Sitter regenerations [Dan11, Dan, Dan13].

A notable feature of the transition constructed here is that it is not constructed with $\mathbb{R}\mathsf{P}^n, \mathbb{C}\mathsf{P}^n$ or any other fixed ambient geometry, and so it is not immediately obvious how to formalize it within the theory of conjugacy limits. Consequently in Section 3 we introduce more flexible formalism for transitional geometry, which will be greatly utilized and expanded upon in the forthcoming sequel [Tre18].

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2 Hyperbolic Space Defined Over Algebras

It is common to define real hyperbolic space constructively via the hyperboloid model in Minkowski space, which we review briefly here. Given a quadratic form q of signature $(n,1)$ on $\mathbb{R}^{n+1}$ the group of orientation preserving isometries of q, denoted $SO(n,1)$ acts transitively on the level sets $X_\alpha = q^{-1}\{\alpha\}$ of q. This defines a subgeometry $(\mathbb{P}X_{-1}, \mathsf{PSO}(n,1))$ of projective space called the *Klein model* of $\mathbb{H}^n_\mathbb{R}$ with $\mathbb{P}X_{-1}$ the interior of an ellipsoid in $\mathbb{R}\mathsf{P}^n$.

In this section we build a collection of three new homogeneous spaces following by analogy the construction of $\mathbb{H}^n_\mathbb{R}$ above, but replacing $\mathbb{R}$ with other real algebras. For the purposes of this paper, it suffices to restrict ourselves to 2-dimensional real algebras, of which there are three up to isomorphism: $\mathbb{C}, \mathbb{R}\oplus\mathbb{R}$ and the so called 'dual numbers' $\mathbb{R}_\varepsilon = \mathbb{R}[\varepsilon]/\varepsilon^2$. We denote the analog of hyperbolic n-space defined over an algebra A by $\mathbb{H}^n_A$.

2.1 $\mathbb{H}^n_{\mathbb{C}}$

The analog of quadratic forms and orthogonal groups over $\mathbb{C}$ are Hermitian forms and unitary groups respectively, defined using complex conjugation $z \mapsto \overline{z}$. The unitary group $\mathsf{U}(n,1;\mathbb{C})$ acts on $\mathbb{C}^{n+1}$ preserving the hermitian form $q = (z,w) = \langle z,w\rangle = \sum_{i=1}^n z_i\overline{w_i} - z_{n+1}\overline{w}_{n+1}$ and acting transitively on each of its level sets, and thus on their projectivizations. Topologically $\mathsf{U}(n,1;\mathbb{C}) = \mathsf{U}(1;\mathbb{C}) \times \mathsf{SU}(n,1;\mathbb{C})$ for $\mathsf{U}(1;\mathbb{C})$ the scalar matrices with entries on the unit circle; as $\mathsf{U}(1;\mathbb{C})$ acts trivially on the projectivized level sets we consider the locally effective action of $\mathsf{SU}(n,1;\mathbb{C})$ for simplicity. Fixing the level set $X_{-1} = q^{-1}\{-1\}$, we define *complex hyperbolic space* to be given by $(\mathsf{SU}(n,1), \mathbb{P}X_{-1})$ as a subgeometry of $\mathbb{C}\mathsf{P}^n$.

To encode this geometry in the group-stabilizer formalism, we need only choose a point x in X_{-1} and determine the stabilizer of its projectivization. The point $x = (0,\ldots,0,1)$ lies in X_{-1} for all n, and thus provides a natural uniform choice.

Definition 2.1. Hyperbolic space over $\mathbb{C}$ is the pair $(\mathsf{SU}(n,1;\mathbb{C}), \mathsf{St}(n,1;\mathbb{C}))$ defined by

$$\mathsf{SU}(n,1;\mathbb{C}) = \left\{A \in \mathsf{GL}(n;\mathbb{C}) \middle| A^\dagger I_{n,1} A = I_{n,1}, \mathsf{det}(A) = 1\right\}$$

$$\mathsf{St}(n,1;\mathbb{C}) = \left\{\begin{pmatrix}\overline{x}B & \vec{0}\\ 0 & x\end{pmatrix} \middle| x \in \mathsf{U}(1;\mathbb{C}), B \in \mathsf{SU}(n;\mathbb{C})\right\}$$

2.2 $\mathbb{H}^n_{\mathbb{R}\oplus\mathbb{R}}$

We view $\mathbb{R}\oplus\mathbb{R}$ as the algebra generated over by $\mathbb{R}$ an additional square root of 1, $\mathbb{R}\oplus\mathbb{R} \cong \mathbb{R}[\lambda]/(\lambda^2 = 1)$. The principal idempotents $e_+ = \frac{1}{2}(1+\lambda)$ and $e_- = \frac{1}{2}(1-\lambda)$ are exchanged under this conjugation. In analogy to the complex case, we say that an element $z = a + \lambda b$ has *real part* a and *imaginary part* b. We denote by $A^\dagger \in \mathsf{M}(n;\mathbb{R}\oplus\mathbb{R})$ the result of element wise conjugation followed by matrix transposition of A.

The form $q(z,w) = \sum_{i=1}^n z_i\overline{w_i} - z_{n+1}\overline{w}_{n+1}$ induces a norm square $z \mapsto q(z,z)$ with real image; the level sets of this form are preserved by the isometries of q in $\mathsf{GL}(n;\mathbb{R}\oplus\mathbb{R})$. It is easy to see these form the *generalized unitary group* $\mathsf{U}(n,1;\mathbb{R}\oplus\mathbb{R}) = \left\{A \in \mathsf{M}(n;\mathbb{R}\oplus\mathbb{R}) \middle| A^\dagger I_{n,1} A = I_{n,1}\right\}$. The sphere of radius -1 with respect to this norm $X_{-1}(\mathbb{R}\oplus\mathbb{R}) = \left\{v \in (\mathbb{R}\oplus\mathbb{R})^n \middle| \|v\|^2_{n,1} = -1\right\}$ is acted on transitively by $\mathsf{SU}(n;\mathbb{R}\oplus\mathbb{R})$, and the quotient by $\mathsf{U}(\mathbb{R}\oplus\mathbb{R})$ forms a projective model for the unitary geometry $\mathbb{P}X_{-1}(\mathbb{R}\oplus\mathbb{R})$. Encoding this in the group-stabilizer formalism requires calculating the projective point-stabilizer of a point X_{-1}, with $(0,\ldots,0,1)$ an obvious candidate.

Definition 2.2. Hyperbolic geometry over $\mathbb{R}\oplus\mathbb{R}$ is defined by the automorphism-stabilizer pair $(\mathsf{SU}(n,1;\mathbb{R}\oplus\mathbb{R}), \mathsf{St}(n,1;\mathbb{R}\oplus\mathbb{R}))$

$$\mathsf{SU}(n,1;\mathbb{R}\oplus\mathbb{R}) = \left\{A \in \mathsf{GL}(n;\mathbb{R}\oplus\mathbb{R}) \middle| A^\dagger) I_{n,1} A = I_{n,1}, \mathsf{det}(A) = 1\right\}$$

$$\mathsf{St}(n,1;\mathbb{R}\oplus\mathbb{R}) = \left\{\begin{pmatrix}\overline{x}B & \vec{0}\\ 0 & x\end{pmatrix} \middle| x \in \mathsf{U}(1;\mathbb{R}\oplus\mathbb{R}), 0, B \in \mathsf{SU}(n;\mathbb{R}\oplus\mathbb{R})\right\}$$

As this homogeneous space does not appear to be treated in the literature, we spend the rest of this section discussing some basic properties. The unitary subgroups of $\mathsf{GL}(n;\mathbb{R}\oplus\mathbb{R})$ share formal similarities with the orthogonal subgroups of $\mathsf{GL}(n;\mathbb{C})$. Most importantly, the notion of signature is not well-defined on similarity classes as the simple computation below shows.

$$\begin{pmatrix}1 & \\ & \lambda\end{pmatrix}^\dagger \begin{pmatrix}1 & \\ & 1\end{pmatrix} \begin{pmatrix}1 & \\ & \lambda\end{pmatrix} = \begin{pmatrix}1 & \\ & -\lambda^2\end{pmatrix} = \begin{pmatrix}1 & \\ & -1\end{pmatrix}$$

As an immediate consequence, all unitary groups over $\mathbb{R}\oplus\mathbb{R}$ are conjugate to one another, and in particular $\mathsf{diag}(I_{n-1},\lambda)$ conjugates $\mathsf{U}(n-1,1;\mathbb{R}\oplus\mathbb{R})$ to $\mathsf{U}(n;\mathbb{R}\oplus\mathbb{R})$. To avoid the proliferation of negative signs in what follows, we will analyze this conjugate model instead.

As a first observation, the level sets of $\sum_i z_i\overline{z_i}$ are cut out in $\mathbb{R}^{2n}$ as $\sum_i x_i^2 - y_i^2$ under the identification $z_i = x_i + \lambda y_i$ so the associated representation of $\mathsf{SU}(n;\mathbb{R}\oplus\mathbb{R})$ has image in $\mathsf{SO}(n,n) \leqslant \mathsf{SL}(2n;\mathbb{R})$. The general linear group itself $\mathsf{GL}(n;\mathbb{R}\oplus\mathbb{R})$ is isomorphic to the direct product $\mathsf{GL}(n;\mathbb{R})\times\mathsf{GL}(n;\mathbb{R})$ via the projections onto $\mathsf{GL}(n;\mathbb{R})$ given by multiplication by the principal idempotents $A \mapsto (Ae_+, Ae_-)$. We may use this decomposition to understand $\mathsf{U}(n;\mathbb{R}\oplus\mathbb{R})$.

Proposition 2.3. *The group* $\mathsf{U}(n;\mathbb{R}\oplus\mathbb{R})$ *is abstractly isomorphic to* $\mathsf{GL}(n;\mathbb{R})$*, and* $\mathsf{SU}(n;\mathbb{R}\oplus\mathbb{R}) \cong \mathsf{SL}(n;\mathbb{R})$.

Proof. Let $A \in \mathsf{U}(n;\mathbb{R}\oplus\mathbb{R})$ and write $A = Xe_+ + Ye_-$ for $X, Y \in \mathsf{GL}(n;\mathbb{R})$. Recalling that conjugation on $\mathbb{R}\oplus\mathbb{R}$ transposes the principal idempotents, we have $A^\dagger A = (X^T e_- + Y^T e_+)(Xe_+ + Ye_-) = Y^T X e_+ + X^T Y e_-$ and expanding $e_\pm$ and equating real and λ-parts of $A^\dagger A = I$ shows $X^T Y = I$. The injection $X \mapsto Xe_+ + X^{-T}e_-$ from $\mathsf{GL}(n;\mathbb{R})$ to $\mathsf{U}(n;\mathbb{R}\oplus\mathbb{R})$ is easily checked to be a homomorphism, and is surjective by the above computation. By the orthogonality of the principal idempotents, $\mathsf{det}(Xe_+ + Ye_-) = \mathsf{det}(X)e_+ + \mathsf{det}(Y)e_-$, the matrices of real determinant necessarily satisfy $\mathsf{det}(X) = \mathsf{det}(Y)$. Applying this to the elements of $\mathsf{SU}(n;\mathbb{R}\oplus\mathbb{R})$ shows $\mathsf{det}(X) = \mathsf{det}(X^{-T}) = \frac{1}{\mathsf{det}(X)}$, thus $\mathsf{det}(X) = 1$. □

It's useful to quickly revisit the point stabilizer with respect to this description. A matrix $A = Xe_+ + \widehat{X}e_-$ projectively stabilizes the vector $u = ve_+ + we_-$ if $Au = \alpha u$ for $\alpha = \beta e_+ + \gamma e_-$ a unit in $\mathbb{R}\oplus\mathbb{R}$. Writing this out, $Xv = \beta v$ and $\widehat{X}w = \gamma w$ so v is an eigenvector of X and w an eigenvector of $\widehat{X}$. In light of this observation, we may construct a completely different model of $\mathbb{H}^n_{\mathbb{R}\oplus\mathbb{R}}$, which will make clear its connection to real projective geometry.

Self Dual Real Projective Space

Given a real vector space V and the dual space of linear functionals $V^\vee$, the natural pairing by evaluation defines a map $\mathsf{ev}\colon V^\vee \times V \to \mathbb{R}$ sending $(\phi, v) \mapsto \phi(v)$. Associated to the action of $\mathsf{GL}(V)$ on V by left multiplication is a left action of $\mathsf{GL}(V)$ on $V^\vee$ by precomposition with the inverse, $(X.\phi)(v) := \phi(X^{-1}v)$. Together this defines an action of $\mathsf{GL}(V)$ on $V^\vee \times V$ which respects the pairing ev. Choosing a basis for V and taking the dual basis for $V^\vee$, the action of $\mathsf{GL}(V)$ on $V^\vee$ corresponds to the inverse transpose automorphism of $\mathsf{GL}(n;\mathbb{R})$, thus $X.(\phi, v) = (\widehat{X}\phi, Xv)$ for $\widehat{X} = X^{-T}$. An easy computation verifies that this action of $\mathsf{SL}(V)$ on $V^\vee \times V$ is transitive on the level sets of ev, and the geometry associated to any two nonzero level sets are isomorphic.

Writing out the pairing ev in a basis and the associated dual for V, $V^\vee$ respectively shows $\mathsf{ev}(\vec{\phi}, \vec{v}) = \sum_i \phi_i v_i$ is a quadratic form of signature (n, n). Thus the associated representation $\mathsf{SL}(V) \mapsto \mathsf{SL}(2n, \mathbb{R})$ has image in (a conjugate of) $\mathsf{SO}(n, n)$. The projectivization $V^\vee \times V \to \mathbb{P}(V^\vee)\times\mathbb{P}(V)$ identifies all nonzero level sets of ev with the same subset of $\mathbb{R}\mathsf{P}^{n-1}\times\mathbb{R}\mathsf{P}^{n-1}$. The zero level set consists of points (ϕ, v) such that $\phi(v) = 0$, and so its projectivization can be interpreted as the set of pointed lines in $\mathbb{R}\mathsf{P}^n$; it is the natural embedding of the flag variety of $\mathbb{R}\mathsf{P}^{n-1}$ into $\mathbb{R}\mathsf{P}^{n-1}\times\mathbb{R}\mathsf{P}^{n-1}$. The complement of the flag variety is an open subset of $\mathbb{R}\mathsf{P}^{n-1}\times\mathbb{R}\mathsf{P}^{n-1}$, realized as the projectivization of the unit sphere of ev. This can be thought of as the space of pairs of a projective line and a point not on that line, with automorphisms the projective transformations of $\mathbb{R}\mathsf{P}^{n-1}$.

We record this geometry in the group-stabilizer formalism by identifying the stabilizer of a point $([\phi], [v])$. As $X.([\phi], [v]) = ([\phi], [v])$ iff ϕ is an eigenvector of $\widehat{X}$ and v is an eigenvector of X. Comparing this with proposition 2.3 and the following remarks gives this important observation.

Observation 2.4. Hyperbolic geometry over $\mathbb{R}\oplus\mathbb{R}$ is isomorphic to self-dual projective geometry.

2.3 $\mathbb{H}^n_{\mathbb{R}_\varepsilon}$

Finally we consider the third isomorphism type of two dimensional algebra, $\mathbb{R}_\varepsilon = \mathbb{R}[\varepsilon]/\varepsilon^2$. The involution $x+\varepsilon y \mapsto x-\varepsilon y$ on $\mathbb{R}_\varepsilon = \mathbb{R}[\varepsilon]/\varepsilon^2$ provides an analog of complex conjugation defining the Hermitian form $q(z,w) = \sum_{i=1}^n z_i\overline{w_i} - z_{n+1}\overline{w}_{n+1}$. This induces a degenerate norm square $\|z\|^2 = q(z,z)$ given in the coordinates $z_i = x_i + \varepsilon y_i$ by $\|z\|^2 = \sum_{i=1}^n x_i^2 - x_{n+1}^2$. As the imaginary components y_i are not noticed by the norm, level sets for $X_\alpha(\mathbb{R}_\varepsilon) = q^{-1}(\alpha)$ are given by $X_\alpha(\mathbb{R}_\varepsilon) = X_\alpha(\mathbb{R}) \times \mathbb{R}^n$. A domain for the associated hyperbolic geometry is the quotient of the -1 level set by the action of $\mathsf{U}(\mathbb{R}_\varepsilon) = \{\pm 1 + \lambda R\}$, denoted $\mathcal{X}_0(p,q) = \mathbb{P}\left\{v \in \mathbb{R}^n_\varepsilon \middle| \|v\|^2_{p,q} = 1\right\}$. We again choose the point $(0,\dots,0,1) \in X_{-1}(\mathbb{R}_\varepsilon)$ to record its projective stabilizer and express this geometry in the group-stabilizer formalism.

Definition 2.5. Hyperbolic geometry over $\mathbb{R}_\varepsilon$ is defined by the automorphism-stabilizer pair $(\mathsf{SU}(n,1;\mathbb{R}_\varepsilon), \mathsf{St}(n,1;\mathbb{R}_\varepsilon))$

$$\mathsf{SU}(n,1;\mathbb{R}_\varepsilon) = \left\{A \in \mathsf{GL}(n;\mathbb{R}_\varepsilon) \middle| A^\dagger) I_{n,1} A = I_{n,1}, \mathsf{det}(A) = 1\right\}$$

$$\mathsf{St}(n,1;\mathbb{R}_\varepsilon) = \left\{ \begin{pmatrix} \overline{x}B & \vec{0} \\ 0 & x \end{pmatrix} \middle| x \in \mathsf{U}(1;\mathbb{R}_\varepsilon), 0, B \in \mathsf{SU}(n;\mathbb{R}_\varepsilon)) \right\}$$

To understand this geometry better we analyze its automorphism group.

Lemma 2.6. *The group $\mathsf{U}(n,1,\Lambda_0)$ is an extension of $\mathsf{O}(n,1,\mathbb{R})$ by the additive group $\mathbb{R}^{n(n+1)/2}$ for $p+q=n$.*

Proof. Let $X+\lambda Y \in \mathsf{U}(n,1,\Lambda_0)$ for $X,Y \in \mathsf{M}(n+1,\mathbb{R})$. Then $(X+\lambda Y)^* I_{n,1}(X+\lambda Y) = (X^T - \lambda Y^T)I_{n,1}(X+\lambda Y) = I_{n,1}$, and expanding using that $\lambda^2 = 0$ gives $X^T I_{n,1} X + \lambda(X^T I_{n,1} Y - Y^T I_{n,1} X) = I_{n,1}$. Equating real and λ-parts gives $X \in \mathsf{O}(p,q,\mathbb{R})$ and $X^T I_{n,1} Y = Y^T I_{n,1} X$. Noting that each of these is the transpose of the other, this says $X^T I_{n,1} Y$ is symmetric.

We notice that as $\delta = 0$ the map $\pi : \mathsf{U}(n,1;\mathbb{R}_\varepsilon) \to \mathsf{O}(n,1,\mathbb{R})$ given by $X+\lambda Y \mapsto X$ is actually a surjective homomorphism: $\pi\left((X+\lambda Y)(Z+\lambda W)\right) = \pi\left(XZ + \lambda(XW+YZ)\right) = XZ = \pi(X+\lambda Y)\pi(Z+\lambda W)$ It remains to investigate $\ker\pi = \{I+\lambda Y \in \mathsf{U}(n,1,\mathbb{R}_\varepsilon)\}$. The condition that $X^T I_{n,1} Y$ is symmetric reduces to the condition that $I_{n,1}Y$ is symmetric,(using that $I_{n,1} = I_{n,1}^{-1}$) we have map from symmetric matrices to $\ker\pi$ given by $S \mapsto I + \lambda I_{n,1} Y$. Thinking of the symmetric matrices as an additive group, this is an injective homomorphism as $Y+Z \mapsto I+\lambda(Y+Z) = (I+\lambda Y)(I+\lambda Z)$. Thus, we have a short exact sequence $0 \to \mathbb{R}^{(n+1)(n+2)/2} \to \mathsf{U}(n,1,\Lambda_\delta) \to \mathsf{O}(n,1,\mathbb{R}) \to 1$ ☐

Corollary 2.7. *The group homomorphism $\mathsf{GL}(n+1;\mathbb{R}_\varepsilon) \to \mathsf{GL}(n+1;\mathbb{R})$ given by dropping the imaginary part induces an epimorphism of geometries $(\mathsf{U}(n,1;\mathbb{R}_\varepsilon), \mathsf{St}(n,1;\mathbb{R}_\varepsilon)) \twoheadrightarrow (\mathsf{SO}(n,1;\mathbb{R}), \mathsf{SO}(n))$ fibering over real6 hyperbolic space $\mathbb{H}^n_{\mathbb{R}} = (\mathsf{SO}(n,1), \mathsf{SO}(n))$.*

3 Families of Algebras, Groups & Geometries

To define a transition between connecting the three geometries discussed in Section 2 we begin by describing a way to connect their algebraic structures via a 1-parameter family of multiplicative structures on the plane. We then use this to create basic examples of 1-parameter families of spaces and groups, motivating the definitions of *families of spaces* and *families of groups* that are necessary to define *families of geometries.*

3.1 The Transition of Algebras

The 1-parameter family of smooth maps $\mu_\delta\colon \mathbb{R}^2 \to \mathbb{R}$, $\mu_\delta((a,b),(c,d)) = (ac+\delta bd, ad+bc)$ defines a 1-parameter family of $\mathbb{R}$-algebra structures on $\mathbb{R}^2$, one for each fixed value of δ. The resulting isomorphism type depends only on the sign of δ, with $(\mathbb{R}^2,\mu_\delta) \cong \mathbb{C}$ for $\delta < 0$, $(\mathbb{R}^2,\mu_0) \cong \mathbb{R}_\varepsilon$ and $(\mathbb{R}^2,\mu_\delta) \cong \mathbb{R}\oplus\mathbb{R}$ for $\delta > 0$. This 1-parameter family of algebras is not locally trivial due to the transition at $\delta = 0$ and so does not form a bundle of algebras in the usual sense. We call such objects *families of algebras* as in algebraic geometry.

Definition 3.1. A family of algebras over $\mathbb{R}$ is a vector bundle $\pi\colon \mathcal{A} \to \mathbb{R}$ together with a section $\mathit{1}\colon \mathbb{R} \to \mathcal{A}$ and smooth map $\mu\colon \mathcal{A}\times_{\mathbb{R}}\mathcal{A} \to \mathcal{A}$ from the fibered product such that the restriction to each fiber is an algebra $(\mathcal{A}_\delta, \mu_\delta)$ with identity $\mathit{1}(\delta)$.

It will often be useful to realize the algebra $(\mathbb{R}^2,\mu_\delta)$ as $\Lambda_\delta = \mathbb{R}[\lambda]/(\lambda^2 = \delta)$. We then denote the transitioning family by $\Lambda_{\mathbb{R}} = \{(x,\delta) | x \in \Lambda_\delta, \delta \in \mathbb{R}\}$ with projection map $\Lambda_{\mathbb{R}} \to \mathbb{R}$ sending $(x,\delta) \mapsto \delta$. The involution defined by $\lambda \mapsto -\lambda$ on $\mathbb{R}[\lambda]$ descends to each Λ_δ, realizing complex conjugation for $\delta < 0$ and swapping the principal idempotents $e_\pm = \frac{1}{2}(1 \pm \frac{\lambda}{\sqrt{\delta}})$ for $\Lambda_\delta \cong \mathbb{R}\oplus\mathbb{R}$ when $\delta > 0$. We denote this involution $x \mapsto \overline{x}$ on each Λ_δ. This induces a multiplicative map $\|\cdot\|_\delta^2\colon \Lambda_\delta \to \mathbb{R}$ given by $x \mapsto x\overline{x}$ such that $\|\cdot\|_{-1}^2$ is the Euclidean norm and $\|\cdot\|_1^2$ is the Minkowski norm $(x,y) \mapsto x^2-y^2$. The units of Λ_δ are the preimage of $\mathbb{R}^\times$ under $\|\cdot\|_\delta$, and the unit sphere in Λ_δ with respect to this square-norm is the one dimensional unitary group $\mathsf{U}(\Lambda_\delta) = \{\alpha \in \Lambda_\delta | \alpha\overline{\alpha} = 1\}$.

For each $n \in \mathbb{N}$ the matrix ring $\mathsf{M}(n,\Lambda_\delta)$ consists of all $n\times n$ matrices over Λ_δ. Matrix multiplication varies continuously depends only on μ_δ and the field operations of $\mathbb{R}$, so these also form a family of algebras $\mathsf{M}(n,\Lambda_{\mathbb{R}}) = \{(A,\delta) | A \in \mathsf{M}(n,\Lambda_\delta)\}$ over $\mathbb{R}$. Conjugation on Λ_δ extends via component-wise application to $\mathsf{M}(n;\Lambda_\delta)$ and induces an involution analogous to the conjugate transpose, $X^\dagger = \overline{X}^T$. The linear action of $\dagger$ on the underlying vector space $\mathsf{M}(n,\Lambda_\delta)$ satisfies $\dagger^2 - 1 = 0$ and decomposes A as a direct sum of the ± 1 eigenspaces, $\mathsf{M}(n;\Lambda_\delta) = \mathsf{Herm}(n) \oplus \mathsf{SkHerm}(n)$ respectively. Note that the definition of $\dagger$ is independent of the algebra multiplication thus Herm and SkHerm do not depend on δ.

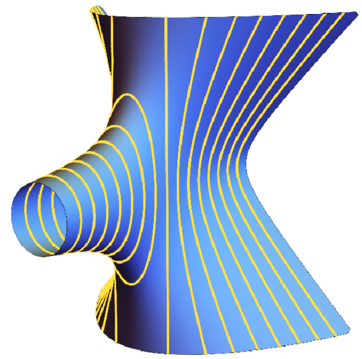
As Λ_δ is commutative, the usual formula for the determinant provides a multiplicative map $\mathsf{det}_\delta\colon \mathsf{M}(n;\Lambda_\delta) \to \Lambda_\delta$. The group of units $\mathsf{GL}(n;\Lambda_\delta) \subset \mathsf{M}(n,\Lambda_\delta)$ is precisely the preimage of $\Lambda_\delta^\times$ under det. The classical groups (special linear, orthogonal, unitary, symplectic) have direct analogues as subgroups of $\mathsf{GL}(n;\Lambda_\delta)$. The special linear group $\mathsf{SL}(n;\Lambda_\delta) = \mathsf{det}_\delta^{-1}\{1\}$ is the subgroup of matrices with determinant 1. The form $I_{p,q} = \mathsf{diag}(I_p, -I_q)$ determines the orthogonal group $\mathsf{O}(p,q;\Lambda_\delta) = \left\{A \in \mathsf{M}(n;\Lambda_\delta) \middle| A^T I_{p,q} A = I_{p,q}\right\}$, with $\mathsf{SO}(p,q;\Lambda_\delta)$ the subgroup of determinant 1. Replacing transpose with the involution $\dagger$ gives the unitary group $\mathsf{U}(p,q;\Lambda_\delta) = \left\{A \in \mathsf{M}(n;\Lambda_delta) \middle| A^\dagger I_{p,q} A = I_{p,q}\right\}$ and its determinant-1 subgroup $\mathsf{SU}(p,q;\Lambda_\delta)$. The (noncompact) symplectic group consists of the matrices preserving the symplectic form $\Omega_n = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$, or $\mathsf{Sp}(2n;\Lambda_\delta) = \left\{A \in \mathsf{M}(2n;\Lambda_\delta) \middle| A^T\Omega_n A = \Omega_n\right\}$; the compact symplectic group is its intersection with the unitary group $\mathsf{U}(I_{2n};\Lambda_\delta)$.

For each δ, $\mathsf{M}(n,\Lambda_\delta)$ acts linearly on Λ_δ^n in the usual way. Identifying Λ_δ with its underlying vector space $\mathbb{R}^2$ gives a linear representation $\iota_\delta\colon \mathsf{M}(n,\Lambda_\delta) \to \mathsf{M}(2n,\mathbb{R})$. When $n = 1$ this is the family of representations $\iota_\delta\colon \Lambda_\delta \to \mathsf{M}(2,\mathbb{R})$ given by $a + \lambda b \mapsto \left(\begin{smallmatrix} a & \delta b \\ b & a \end{smallmatrix}\right)$ generalizing the standard 2-dimensional representation of $\mathbb{C}$.

3.2 Families of Spaces, Groups and Geometries

To motivate the definitions, we consider the unit spheres $\mathsf{U}(\Lambda_{\mathbb{R}}) = \left\{(\alpha,\delta) \middle| \|\alpha\|_\delta^2 = 1\right\}$ in the family $\Lambda_{\mathbb{R}}$. The norm square $\Lambda_\delta \to \mathbb{R}$ is built out of multiplication and inversion and defined by a polynomial map $\|x + \lambda y\|_\delta^2 = x^2 - \delta y^2$, which identifies $\mathsf{U}(\Lambda_{\mathbb{R}})$ with the smooth irreducible real algebraic variety $V(x^2 - \delta y^2 - 1) \subset \mathbb{R}^3$. The family projection $\pi\colon \Lambda_{\mathbb{R}} \to \mathbb{R}$ restricts to a smooth submersion $\mathsf{U}(\Lambda_{\mathbb{R}}) \to \mathbb{R}$ with point preimages $\pi|_{\mathsf{U}(\Lambda_{\mathbb{R}})}^{-1}\{\delta\} = \mathsf{U}(\Lambda_\delta)$ the individual unit spheres.

These slices vary from the round circle $\mathsf{U}(\Lambda_{-1})$ through ellipses of increasing eccentricity to a pair of vertical lines $\mathsf{U}(\Lambda_0)$ before varying through hyperboloids $\mathsf{U}(\Lambda_\delta)$ for $\delta > 0$. We call a smooth manifold foliated by submanifolds a *family* when the leaves of the foliation are point preimages of a submersion, as here.



Definition 3.2. A family of spaces is a smooth manifold $\mathcal{X}$ equipped with a submersion $\pi\colon \mathcal{X} \to \mathbb{R}$. The space $\mathcal{X}$ is called the total space of the family, and the point preimages $\mathcal{X}_\delta := \pi^{-1}\{\delta\}$ are the *members*.

The algebraic structure of $\Lambda_\mathbb{R}$ provides the family of unit spheres with even more structure. As the norm-square is multiplicative each $\mathsf{U}(\Lambda_\delta)$ inherits a group structure from $\Lambda_\delta^\times$ with identity $1 \in \mathbb{R} \subset \Lambda_\delta$, and multiplication (resp. inversion) given by restrictions of μ_δ (resp. conjugation). Since the maps μ, $\bar{\cdot}$ restrict to smooth maps on all of $\mathsf{U}(\Lambda_\mathbb{R}) \times_\mathbb{R} \mathsf{U}(\Lambda_\mathbb{R})$ and $\mathsf{U}(\Lambda_\mathbb{R})$ respectively, we say the group operation on $\mathsf{U}(\Lambda_\delta)$ varies smoothly with δ. This provides the motivating example for defining families of groups.

Definition 3.3. A *family of groups* is a family of spaces $\pi\colon \mathcal{G} \to \mathbb{R}$ together with a smooth section $e\colon \mathbb{R} \to \mathcal{G}$ and smooth maps $\mu\colon \mathcal{G} \times_\mathbb{R} \mathcal{G} \to \mathcal{G}$, $\iota\colon \mathcal{G} \to \mathcal{G}$ such that for each δ, the space $\mathcal{G}_\delta$ is a group with identity $e(\delta)$, multiplication $\mu|_{\mathcal{G}_\delta \times \mathcal{G}_\delta}$ and inversion $\iota|_{\mathcal{G}_\delta}$.

A family of groups is an example of a Lie groupoid (a groupoid object in the category of smooth manifolds) with equal source and target maps into $\mathbb{R}$. Such Lie groupoids have been employed in describing transitional behavior in differential geometry under the name *bundles of groups* such as in [GBPS14]. Geometries are given by transitive actions of groups on spaces. In practice it is often much easier to work with the group-stabilizer formalism, and so we develop families of geometries from this viewpoint.

Definition 3.4. A family of geometries is given by a pair $(\mathcal{G}, \mathcal{K})$ of families of groups such that for each $\delta \in \mathbb{R}$, $\mathcal{K}_\delta \leqslant \mathcal{G}_\delta$ is a Lie subgroup.

The domains for these geometries may be abstractly identified with the coset spaces $\mathcal{X}_\delta = \mathcal{G}_\delta/\mathcal{K}_\delta$, with the action of $\mathcal{G}_\delta$ on $\mathcal{X}_\delta$ given by translation. We do not pursue this here, but one can show that given a family $(\mathcal{G}, \mathcal{K})$ of geometries as above, there is a unique smooth structure on the quotient space $\mathcal{X} = \mathcal{G}/\mathcal{K} = \bigcup_{\delta \in \mathbb{R}} \mathcal{G}_\delta/\mathcal{K}_\delta$ such that the induced projection $\pi\colon \mathcal{X} \to \mathbb{R}$ gives $\mathcal{X}$ the structure of a smooth family of spaces with a fiberwise transitive action of the family of groups $\mathcal{G}$.

A morphism of families over the same base is a fiber preserving smooth map $\phi\colon \mathcal{X} \to \mathcal{X}$ (that is, $\pi_\mathcal{Y}\phi = \pi_\mathcal{X}$ as maps into $\mathbb{R}$). A homomorphism of families of groups is a morphism $\mathcal{G} \to \mathcal{H}$ of their underlying spaces which restricts fiberwise to a group homomorphism. A morphism of families of geometries is a homomorphism of pairs $(\mathcal{G}, \mathcal{K}) \to (\mathcal{H}, \mathcal{C})$. Injective morphisms $\mathcal{F}_1 \hookrightarrow \mathcal{F}_2$ of families are called embeddings, and a family $\mathcal{F}_1$ is said to fiber over $\mathcal{F}_2$ if there is a surjective morphism $\mathcal{F}_1 \to \mathcal{F}_2$.

A family $\mathcal{F} \to \mathbb{R}$ is called trivial if the isomorphism type of the members is constant; it is called nontrivial otherwise. For the purposes of this article, the most important families are the simplest nontrivial examples; the transitions.

Definition 3.5. A family $\mathcal{F} \to \mathbb{R}$ is a transition from U to V through W if $\mathcal{F}_\delta \cong U$ for $\delta < 0$, $\mathcal{F}_\delta \cong V$ for $\delta > 0$ and $\mathcal{F}_0 \cong W$.

3.3 The $\mathbb{H}^2 \leftrightarrow \mathbb{E}^2 \leftrightarrow \mathbb{S}^2$ Transition

We illustrate this by formalizing the well-known transition from hyperbolic to spherical geometry through Euclidean.

Let q_δ be the quadratic form on $\mathbb{R}^3$ defined by $q_\delta(x,y,z)=\delta(x^2+y^2)+z^2$. The orientation-preserving isometries of q_δ form the group $\mathsf{PSO}(q_\delta)\leqslant\mathsf{PGL}(3;\mathbb{R})$, which has signature $(1,2)$ for $\delta<0$ and signature $(3,0)$ for $\delta>0$. The unit spheres for q_δ are the (double covers of the) domains for the geometries; together they form a family of spaces cut out of $\mathbb{R}^3\times\mathbb{R}$ via the polynomial $\mathcal{S}_\mathbb{R}=V(\delta(x^2+y^2)+z^2-1)$ equipped with the projection $(x,y,z,\delta)\mapsto\delta$.

Let $\mathcal{G}\subset\mathsf{PGL}(3;\mathbb{R})\times\mathbb{R}$ consist of the groups $\mathcal{G}_\delta=\mathsf{PSO}(q_\delta)$ for each $\delta\neq 0$ and $\mathcal{G}_0=\mathsf{Euc}(2)=\left\{\begin{pmatrix}\mathsf{SO}(2)&\mathbb{R}^2\\0&1\end{pmatrix}\right\}$ be the Euclidean group. Then for each δ the pair $(\mathcal{G}_\delta,\mathbb{P}\mathcal{S}_\delta)$ is a geometry in the sense of Klein, isomorphic to $\mathbb{H}^2$ for $\delta<0$, $\mathbb{E}^2$ for $\delta=0$ and elliptic space for $\delta>0$. The point $(0,0,1)$ lies on $\mathcal{S}_\delta$ for each δ, and so we let $\mathcal{K}_\delta=\mathsf{stab}_{\mathcal{G}_\delta}[0:0:1]$. To formalize this transition then we must show that $\mathcal{G}$ and $\mathcal{K}$ are families of groups.

The case of $\mathcal{K}$ is straightforward; for each $\delta\in\mathbb{R}$ the stabilizer is $\mathcal{K}_\delta=\left\{\begin{pmatrix}\mathsf{SO}(2)&0\\0&1\end{pmatrix}\right\}$ thus $\mathcal{K}$ is a trivial family over $\mathbb{R}$. Understanding $\mathcal{G}$ is more subtle as $\mathcal{G}_0\neq\mathsf{PSO}(q_0)$ (with two degenerate eigenvalues, there are many more isometries of q_0). The groups $\mathcal{G}_\delta$ do admit a nice uniform description as $\mathcal{G}_\delta=\widehat{\mathsf{PSO}(J_\delta)}$ for $J_\delta=\mathsf{diag}(1,1,\delta)$ and $\widehat{\cdot}$ the contragredient however [2]. Due to the degeneracy of only one eigenvalue of the form as $\delta\to 0$, the collection $\mathcal{H}=\{(A,\delta)|A\in\mathsf{SO}(J_\delta)\}$ is a smooth submanifold of $\mathsf{GL}(3;\mathbb{R})\times\mathbb{R}$ cut out by polynomials $V(A^TJ_\delta A=J_\delta,\mathsf{det}(A)=1)$. Together with the projection $(A,\delta)\mapsto\delta$ this is a smooth family of groups. Fiberwise applying the contragredient is a smooth automorphism of the family $\mathsf{GL}(3;\mathbb{R})\times\mathbb{R}$, and so the image $\widehat{\mathcal{H}}=\mathcal{G}$ is a family of groups as well.

4 The Families of Groups $\mathsf{SU}(p,q;\Lambda_\mathbb{R})$

The main work in constructing a transition of geometries from the algebras $\Lambda_\mathbb{R}$ is showing that the automorphism groups $\mathsf{SU}(n,1;\Lambda_\delta)$ vary continuously with δ. As a first step towards this, we realize the individual unitary groups as point preimages under a submersion; the argument proceeds in direct analogy to the standard case over $\mathbb{C}$.

Lemma 4.1. *Let $J=\mathsf{diag}(I_p,-I_q)$. The map $\Phi_\delta\colon\mathsf{M}(n;\Lambda_\delta)\to\mathsf{Herm}(n;\Lambda_\delta)$ given by $\Phi_\delta(A)=A^\dagger JA$ is a smooth submersion.*

Proof. Let $B\in\mathsf{U}(J)$, then for any $X\in\mathsf{M}(n,\Lambda_\delta)$ we may construct the path $B_t=B+tX$ which remains in $\mathsf{GL}(n,\Lambda_\delta)$ for small t. Computing the derivative we see $\frac{\mathsf{d}}{\mathsf{dt}}|_{t=0}\Phi_\delta(B_t)=X^\dagger JB+B^\dagger JX$, and so Φ_J is a submersion if $X\mapsto X^\dagger JB+B^\dagger JX$ surjects onto $T_{\Phi_\delta(B)}\mathsf{Herm}(n)=\mathsf{Herm}(n)$. This map is $\mathbb{R}$-linear and so we proceed by dimension count, noting $\dim\operatorname{image}\Phi_\delta=\dim\mathsf{M}(n,\Lambda_\delta)-\dim\ker\Phi_\delta$. The kernel of Φ_δ is given by $\ker\Phi_\delta=\left\{X\middle|X^\dagger JB=-B^\dagger JX\right\}$, which can be expressed $\ker\Phi_\delta=(B^\dagger J)^{-1}\mathsf{SkHerm}(n)$. Thus $\dim\ker\Phi_\delta$ is the dimension of the space of skew-Hermitian matrices, so $\dim\operatorname{image}\Phi_\delta=\dim\mathsf{Herm}(n)$ and $(D\Phi_\delta)_B$ is surjective, making Φ_δ is a submersion. ☐

Corollary 4.2. *The unitary groups $\mathsf{U}(p,q;\Lambda_\delta)$ are smooth submanifolds of $\mathsf{M}(p+q;\Lambda_\delta)$ by the preimage theorem as $\mathsf{U}(p,q;\Lambda_\delta)=\Phi_\delta^{-1}\{J\}$.*

Lemma 4.3. *The determinant restricts to a submersion* $\mathsf{det}\colon\mathsf{U}(p,q;\Lambda_\delta)\to\mathsf{U}(\Lambda_\delta)$.

Proof. The defining condition of $\mathsf{U}(p,q;\Lambda_\delta)$ shows $\mathsf{det}|_{\mathsf{U}(p,q;\Lambda_\delta)}$ takes values in $\mathsf{U}(\Lambda_\delta)$, thus defining the short exact sequence $1\to\mathsf{SU}(p,q;\Lambda_\delta)\to\mathsf{U}(p,q;\Lambda_\delta)\to\mathsf{U}(\Lambda_\delta)\to 1$. This is right-split by the section $\alpha\mapsto\mathsf{diag}(\alpha,1,\ldots,1)$ so $\mathsf{U}(p,q;\Lambda_\delta)$ is topologically a product $\mathsf{U}(\Lambda_\delta)\times\mathsf{SU}(p,q;\Lambda_\delta)$. Under these coordinates the determinant is a projection, thus a smooth submersion. ☐

[2] For $\delta\neq 0$ this follows easily from the observation $\widehat{\mathsf{O}([J])}=\mathsf{O}([J^{-1}])$ for nondegenerate J, and is a quick computation for $\delta=0$.

In particular this shows $\mathsf{SU}(p,q;\Lambda_\delta)$ is a smooth submanifold of $\mathsf{U}(p,q;\Lambda_\delta)$ (this was already clear from the closed subgroup theorem, but we have other use for lemma 4.3 in the future). To make a similar argument on the level of families, we piece together the maps Φ_δ into a map $\Phi\colon \mathsf{M}(n;\Lambda_\mathbb{R}) \to \mathsf{M}(n;\Lambda_\mathbb{R})$ with $\Phi(A,\delta) = (\Phi_\delta(A),\delta)$. This map is easily seen to be smooth as it is constructed out of the operations inherent to the algebra family $\mathsf{M}(n;\Lambda_\mathbb{R})$. The following observation and lemma allow us to show Φ is a submersion by checking that its differential is onto each subspace in the decomposition; as this is the content of the lemma above, it essentially completes the proof.

Observation 4.4. Let $\sigma : \mathbb{R} \to \mathcal{X}$ be a smooth section of a submersion $\pi\colon \mathcal{X} \to \mathbb{R}$. Then for each $x = \sigma(\delta)$ the tangent space $T_x\mathcal{X}$ decomposes as a direct sum $T_x\mathcal{X} = T_x\sigma(\mathbb{R}) \oplus T_x\pi^{-1}\{\delta\}$ into 'vertical' and 'horizontal' factors.

Lemma 4.5. *Let $\Phi\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of smooth families. Then Φ is a submersion if for each $\delta \in \mathbb{R}$ the restriction $\Phi_\delta\colon \mathcal{X}_\delta \to \mathcal{Y}_\delta$ is a submersion.*

Proof. Let $x \in \mathcal{X}$ with $\pi_\mathcal{X}(x) = \delta$ and choose a section $\sigma\colon U \to \mathcal{X}$ through x [3]. Then $\Phi\sigma\colon U \to \mathcal{Y}$ is a section through $y = \Phi(x)$, and so by the observation above σ and $\Phi\sigma$ provide the direct sum decompositions $T_x\mathcal{X} = T_x\sigma(U) \oplus T_x\mathcal{X}_\delta$ and $T_y\mathcal{Y} = T_y\Phi\sigma(U) \oplus T_y\mathcal{Y}_\delta$. Restricting Φ to $\sigma(U)$ gives a homeomorphism $\sigma(U) \to \Phi\sigma(U)$ so $d\Phi_x|_{T_x\sigma(U)}$ is an isomorphism onto $T_y\Phi\sigma(U)$. But by assumption the restriction $\Phi_\delta\colon \mathcal{X}_\delta \to \mathcal{Y}_\delta$ is a submersion, thus $d\Phi_x|_{T_x\mathcal{X}_\delta}$ maps onto $T_y\mathcal{Y}_\delta$ so all together $d\Phi_x\colon T_x\mathcal{X} \to T_y\mathcal{Y}$ is surjective and Φ is a submersion. □

Corollary 4.6. *The map $\Phi\colon \mathsf{M}(n;\Lambda_\mathbb{R}) \mapsto \mathsf{Herm}(n;\Lambda_\mathbb{R})$ defined by $(A,\delta) \mapsto (A^\dagger JA,\delta)$ is a smooth submersion.*

This follows directly as Φ is a morphism of families which fiberwise gives a submersion $\Phi_\delta\colon \mathsf{M}(n;\Lambda_\delta) \to \mathsf{Herm}(n)$. Given this, $\mathsf{U}(p,q;\Lambda_\mathbb{R}) = \Phi^{-1}\{J \times \mathbb{R}\}$ is a smooth manifold of $\mathsf{M}(n,\Lambda_\mathbb{R})$. Restricting the determinant to this gives a smooth map $\mathsf{U}(p,q;\Lambda_\mathbb{R}) \to \mathsf{U}(\Lambda_\mathbb{R})$ which is fiberwise a submersion by lemma 4.3, so $\mathsf{SU}(p,q;\Lambda_\delta) = \mathsf{det}^{-1}\{1 \times \mathbb{R}\}$ is a smooth submanifold of $\mathsf{U}(p,q;\Lambda_\mathbb{R})$. We are now in a position to show that these are families of groups, as claimed.

Proposition 4.7. *Equipped with the restriction of $\pi\colon \mathsf{M}(n;\Lambda_\mathbb{R}) \to \mathbb{R}$, the collections $\mathsf{U}(p,q;\Lambda_\mathbb{R})$ and $\mathsf{SU}(p,q;\Lambda_\mathbb{R})$ form smooth families of groups.*

Proof. We may realize $\mathsf{U}(p,q;\Lambda_\mathbb{R})$ as the pullback of the morphisms $\Phi\colon \mathsf{M}(n;\Lambda_\mathbb{R})$ and the constant $\mathcal{J}\colon \mathbb{R} \to \mathsf{Herm}(\Lambda_\mathbb{R})$ defined by $\mathcal{J}(\delta) = (I_{p,q},\delta)$.

$$\begin{array}{ccc}
\mathsf{SU}(n,1;\Lambda_\mathbb{R}) & \longrightarrow & \mathsf{GL}(n+1;\Lambda_\mathbb{R}) \\
\big\downarrow & & \big\downarrow{\scriptstyle \Phi} \\
\mathbb{R} & \xrightarrow{\ \mathcal{J}\ } & \mathsf{Herm}(n+1;\Lambda_\mathbb{R})
\end{array}$$

Recalling that submersions are characterized by admitting smooth sections through each point of the domain, it suffices to show that there is a local section of $\pi\colon \mathsf{U}(p,q;\Lambda_\mathbb{R}) \to \mathbb{R}$ through each $(A,\delta) \in \mathsf{U}(p,q;\Lambda_\mathbb{R})$. Let (A,δ) be such a point. Then as $\Phi\colon \mathsf{M}(n;\Lambda_\mathbb{R}) \to \mathsf{Herm}(\Lambda_\mathbb{R})$ is a smooth submersion by lemma 4.1, we may choose a local section $\sigma\colon U \to \mathsf{M}(n;\Lambda_\mathbb{R})$ of Φ with $(A,\delta) = \sigma(J,\delta)$. Precomposing with $\mathcal{J}$ pulls this back to $\tau = \sigma \circ \mathcal{J}\colon W \to \mathsf{M}(n;\mathbb{R})$ for $W = \mathcal{J}^{-1}(U) \subset \mathbb{R}$ an open set containing δ. Clearly τ is a section of π as all maps involved are fiber preserving, and $\tau(\delta) = \sigma(J,\delta) = (A,\delta)$. Thus it remains only to see that $\tau(W) \subset \mathsf{U}(p,q;\Lambda_\mathbb{R})$. But this is similarly clear; for any $t \in W$, $\tau(t) = \sigma(I_{p,q},t)$ and as σ is a right inverse to Φ, $\sigma(J,t)^\dagger I_{p,q}\sigma(I_{p,q},t) = I_{p,q}$, so $\sigma(J,t) \in \mathsf{U}(p,q;\Lambda_t)$. To conclude the result for $\mathsf{SU}(p,q;\Lambda_\mathbb{R})$ we realize it as the pullback of $\mathsf{det}\colon \mathsf{U}(p,q;\Lambda_\mathbb{R}) \to \mathsf{U}(\Lambda_\mathbb{R})$ along $\mathit{1}\colon \mathbb{R} \to \mathsf{U}(\Lambda_\delta)$ defined by $t \mapsto (1,t)$, and perform a similar analysis. □

[3] a map is a smooth submersion if and only if it admits smooth sections through each point of the domain

5 The Transition $\mathbb{H}^n_{\Lambda_\delta}$

Using the notion of family introduced above, in this section we construct a transition of geometries connecting complex hyperbolic space to self-dual real projective space.

Definition 5.1. The family of hyperbolic geometries over $\Lambda_\mathbb{R}$ is given by the automorphism-stabilizer pair $(\mathsf{SU}(n,1;\Lambda_\mathbb{R}), \mathsf{St}(n,1;\Lambda_\mathbb{R}))$ for

$$\mathsf{SU}(n,1;\Lambda_\mathbb{R}) = \left\{(A,\delta)\middle| A\in\mathsf{M}(n;\Lambda_\delta), A^\dagger I_{n,1}A = I_{n,1}\right\}$$

$$\mathsf{St}(n,1;\Lambda_\mathbb{R}) = \left\{\left(\left(\begin{smallmatrix}\overline{x}B & \vec{0}\\ \vec{v} & x\end{smallmatrix}\right),\delta\right)\middle| x\in\mathsf{U}(1;\Lambda_\delta), \vec{v}\in\Lambda_\delta^{n-1}, B\in\mathsf{SU}(n;\Lambda_\delta)\right\}$$

The domain of this geometry can be constructed from the linear action on Λ_δ^n in analogy to the constructions over $\mathbb{C},\mathbb{R}_\varepsilon$ and $\mathbb{R}\oplus\mathbb{R}$ in Section 2. We denote by $\mathcal{X}_{-1}(\delta)$ the projectivization by $\mathsf{U}(\Lambda_\delta)$ of the -1 level set for $||\cdot\|_\delta$ on Λ_δ^n, $\mathcal{X}_{-1}(\delta) = \mathbb{P}\left\{v\in\Lambda_\delta^n\middle|\|v\|^n_{p,q}=1\right\}$. That this collection of geometries forms a family is an easy consequence of the work in Section 4, where specializing to $(p,q)=(n,1)$ gives the result for the automorphism groups. Here we see that the continuity of the stabilizer family follows directly from the case $(p,q)=(n,0)$.

Proposition 5.2. *The stabilizer family* $\mathsf{St}(n,1;\Lambda_\mathbb{R})$ *is a smooth family of groups over* $\mathbb{R}$ *equipped with the restricted projection* $\mathsf{M}(n+1;\Lambda_\mathbb{R})\to\mathbb{R}$*.*

Proof. Recall the form of matrices in the stabilizing subgroups are $\left(\begin{smallmatrix}\overline{x}A & \vec{0}\\ \vec{v} & x\end{smallmatrix}\right)$ with $x\in\mathsf{U}(1;\Lambda_\delta)$, $\vec{v}\in\Lambda_\delta^{n-1}$ and $A\in U(n;\Lambda_\delta)$. As δ varies this gives a subset of $\mathsf{M}(n+1;\Lambda_\mathbb{R})$ which is topologically a product of $\Lambda_\mathbb{R}^n$ with the collection $\{(\left(\begin{smallmatrix}\overline{x}A & 0\\ 0 & x\end{smallmatrix}\right),\delta)|A\in\mathsf{SU}(n;\Lambda_\mathbb{R}), x\in\mathsf{U}(1;\Lambda_\mathbb{R})\}$.

But, the work of Section 4, $\mathsf{SU}(n;\Lambda_\mathbb{R})$ and $\mathsf{U}(1;\Lambda_\mathbb{R})$ are smooth families over $\mathbb{R}$ and so is their product. The embedding $\mathsf{U}(1;\Lambda_\mathbb{R})\times\mathsf{SU}(n;\Lambda_\mathbb{R})\to\mathsf{M}(n+1;\Lambda_R)$ given by $((x,A),\delta)\mapsto(\left(\begin{smallmatrix}x & 0\\ 0 & \overline{x}A\end{smallmatrix}\right),\delta)$ is a smooth embedding of families, so the image is a family and thus the stabilizers themselves form a family. □

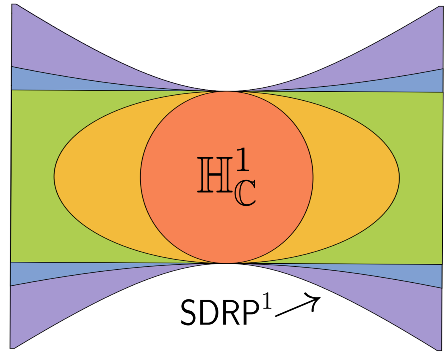
The existence of this family proves the following theorem.

Theorem 5.3. *The collection* $(\mathsf{SU}(n,1;\Lambda_\mathbb{R}),\mathsf{St}(n,1;\Lambda_\mathbb{R}))$ *of definition 5.1 is a smooth family of geometries , transitioning from complex hyperbolic space to self dual real projective space through the transitional geometry fibering over* $\mathbb{H}^n$*.*

To understand this transition, it is instructive to look at the domains in $\mathbb{P}(\Lambda_\delta^{n+1})$ preserved by $\mathsf{SU}(1,n;\Lambda_\delta)$. For $\delta<0$, each point in the positive cone for the signature $(1,n)$ form has first coordinate of nonzero norm, and thus up to an element of $\Lambda_\delta^\times$ we may take a representative with first coordinate 1. Thus the entire domain embeds in the affine patch $z_1=1$, and is given by the equation $1>\sum_{i=2}^{n+1}z_i\overline{z_i}$. Writing out $z_i=x_i+\lambda y_i$ and expanding using the multiplication of Λ_δ gives an equation for an ellipsoid with n principal axes of length 1 and n of length $1/\sqrt{|\delta|}$. Thus as $\delta\to0$, this family of ellipsoids converges to $\mathbb{B}^n\times\mathbb{R}^n$.

When $\delta=0$, it is again true that each point in the positive cone of the $(n,1)$ form has a representative with $z_1=1$ (up to elements of $\Lambda_0^\times$) and so the entire domain embeds in the $z_1=1$ patch. Writing down the form it is immediate that this domain is the same $\mathbb{B}^n\times\mathbb{R}^n$ approached above, as expected.

When $\delta > 0$, it is no longer true that each point of the positive cone of the $(n, 1)$ form has first coordinate a unit (for instance, $(0, \lambda)$ has norm 1 under $z_1\overline{z_1} - z_2\overline{z_2}$ in $\mathbb{R} \oplus \mathbb{R}$) so the domain has ideal points with respect to affine models. In the affine patch $z_1 = 1$ the finite points of the domain lie in the interior of a hyperboloid defined by $1 \geq \sum_{i=2}^{n+1} x_i^2 - \delta y_i^2$. This transition on the level of domains in an affine patch is depicted to the right for the case $n = 1$, giving a degeneration of the Poincare disk model of $\mathbb{H}^1_{\mathbb{C}} \cong \mathbb{H}^2_{\mathbb{R}}$.



An important aspect of this transition is that the embedding of $\mathbb{R} \hookrightarrow \Lambda_\delta$ for each δ gives a copy of $\mathsf{SO}(n, 1; \mathbb{R}) \leqslant \mathsf{SU}(n, 1; \Lambda_\delta)$. From the models above we see that in fact there is a unit n-ball slice (the real points) of each affine model which is preserved by this group action, thus the degeneration of complex hyperbolic space occurs while *leaving real hyperbolic space fixed* throughout the entire process.

Theorem 5.4. *The trivial family of real hyperbolic geometries $\mathbb{H}^n \times \mathbb{R} \to \mathbb{R}$ embeds into the transition of theorem 5.3.*

This makes this transition particularly attractive for studying families of complex hyperbolic structures which collapse onto $\mathbb{H}^n_{\mathbb{R}}$.

6 Application: Triangle Groups

The transitional geometry and its automorphisms bridge the gap from complex hyperbolic to self-dual real projective space. Thus understanding the theory of group representations into $\mathsf{SU}(n, 1; \Lambda_0)$ is instrumental in producing transitioning structures. Here we focus on the case of triangle groups and $\mathsf{SU}(2, 1; \Lambda_0)$ in the interest of quick examples, but this is true in much greater generality.

6.1 Generalities

Given a finitely presented group Γ and a Lie group G, the representation variety $\mathrm{Hom}(\Gamma, G)$ consists of all homomorphisms $\Gamma \to G$. Fixing a presentation $\Gamma = \langle g_1, \dots g_n | \; r_1, \dots r_m \rangle$ determines an evaluation map $\mathsf{ev} \colon \mathrm{Hom}(\Gamma, G) \to G^n$ by $\rho \mapsto (\rho(g_1), \dots \rho(g_n))$ which we use to identify $\mathrm{Hom}(\Gamma, G)$ with its image. Local deformations of a representation $\rho \in \mathrm{Hom}(\Gamma, G)$ are understood through the Zariski tangent space $T_\rho \mathrm{Hom}(\Gamma, G)$ which encodes the group relations to first order. Under the evaluation map the Zariski tangent bundle $T \mathrm{Hom}(\Gamma, G)$ identifies with a subset of $(TG)^n$, or $G^n \times \mathfrak{g}^n$ under the natural identifications $T_A G \cong \mathfrak{g}$ via translation.

A triangle Δ with angles $\pi/p, \pi/q, \pi/r$ can be geodesically realized in $\mathbb{H}^2$ if $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$; reflections in the sides generate a discrete subgroup of $\mathsf{Isom}(\mathbb{H}^2)$. We denote the index-two subgroup of orientation preserving transformations by $\Delta(p, q, r)$. The quotient $\mathbb{H}^2/\Delta(p, q, r)$ is an orbifold with underlying space $\mathbb{S}^2$ and three cone points with angles p, q and r submultiples of 2π. Basic orbifold covering space theory gives the presentation $\Delta(p, q, r) = \langle a, b, c | \; a^p + b^q + c^r = abc = 1 \rangle$, from which c can be eliminated to give $\langle a, b | \; a^p = b^q = (ab)^r = 1 \rangle$ where convenient. As triangles are determined up to congruence by their angles in $\mathbb{H}^2$, this hyperbolic structure is rigid and so all discrete & faithful representations $\Delta(p, q, r) \to \mathsf{SO}(2, 1; \mathbb{R})$ are conjugate.

6.2 The limit $\mathsf{U}(n,1;\mathbb{R}_\varepsilon)$

The inner product $\langle X,Y\rangle = \mathsf{tr}(XY^\dagger)$ on $\mathsf{M}(n+1,\mathbb{C})$ restricts to inner products on $\mathfrak{sl}(n+1;\mathbb{R})$ and $\mathfrak{su}(n,1;\mathbb{C})$ for which the orthogonal complements of $\mathfrak{so}(n,1)$ are $N=\{X|\mathsf{tr}(X)=0, JX\in\mathsf{Sym}(n+1;\mathbb{R})\}$ and iN respectively. Moving around $\mathsf{SO}(n,1)$ by left translation, this defines the normal bundles to $\mathsf{SO}(n,1;\mathbb{R})\subset\mathsf{GL}(n+1;\mathbb{R})$ and $\mathsf{SO}(n,1;\mathbb{R})\subset\mathsf{SU}(n,1;\mathbb{C})$ respectively.

Observation 6.1. The normal bundle to $\mathsf{SO}(n,1)\subset\mathsf{SL}(n+1;\mathbb{R})$ is defined by the following: $N_{\mathsf{SL}}=\left\{(A,X)\middle|A\in\mathsf{SO}(n,1),\mathsf{tr}(A^{-1}X)=0, JA^{-1}X\in\mathsf{Sym}(n+1;\mathbb{R})\right\}$. Rotating the tangent direction by i gives the normal bundle N_{SU} to $\mathsf{SO}(n,1)\subset\mathsf{SU}(n,1;\mathbb{C})$.

Recalling the description of lemma 2.6, the group $\mathsf{U}(n,1;\Lambda_0)$ consists of elements $A(I+\lambda U)$ for $A\in\mathsf{SO}(n,1;\mathbb{R})$ and JU symmetric. Thus the homomorphism $\mathsf{SU}(n,1;\Lambda_0)\to\mathsf{SO}(n,1;\mathbb{R})$ projecting onto the real part extends to a homeomorphism onto the normal bundle.

Observation 6.2. Let $\pi_k\colon\mathsf{M}(n+1;\mathbb{R}\oplus\lambda\mathbb{R})\to\mathsf{M}(n+1;\mathbb{R})$ be the projections onto the real and λ factors for $k\in\{1,2\}$ respectively. Then the maps

$$p_{\mathsf{SL}}=(\pi_1,\pi_2)\colon\mathsf{SU}(n,1;\Lambda_0)\to N_{\mathsf{SL}}\qquad p_{\mathsf{SU}}=(\pi_1,i\pi_2)\colon\mathsf{SU}(n,1;\Lambda_0)\to N_{\mathsf{SU}}$$

are homeomorphisms onto their images.

This allows us to view elements of $\mathsf{SU}(n,1;\Lambda_0)$ as recording infinitesimal paths in $\mathsf{SL}(n+1;\mathbb{R})$ or $\mathsf{SU}(n,1;\mathbb{C})$ passing through $\mathsf{SO}(n,1;\mathbb{R})$ orthogonally. This extends more generally to representation varieties themselves, allowing the interpretation of elements of $\mathrm{Hom}(\Gamma,\mathsf{SU}(n,1;\Lambda_0))$ as certain infinitesimal paths in either the $\mathsf{SL}(n+1;\mathbb{R})$ or $\mathsf{SU}(n,1;\mathbb{R})$ representation varieties. We illustrate this with the simple example of representations of $\mathbb{Z}_n$.

Example 6.3. The projections p_{GL} and p_{U} of observation 6.2 identify the representation variety $\mathrm{Hom}(\mathbb{Z}_n,\mathsf{SU}(2,1;\Lambda_0))$ with $T\,\mathrm{Hom}(\mathbb{Z}_m,\mathsf{SL}(n+1;\mathbb{R}))\cap N_{\mathsf{SL}}$ and $T\,\mathrm{Hom}(\mathbb{Z}_m,\mathsf{SU}(n,1;\mathbb{C}))\cap N_{\mathsf{SU}}$ respectively.

Proof. An element of order m in $\mathsf{SU}(n,1;\Lambda_0)$ satisfies $(A+\lambda X)^n=I$; expanding using $\lambda^2=0$ gives $A^m+\lambda(XA^{m-1}+AXA^{m-2}+\ldots+A^{n-1}X)$. Equating real and imaginary parts gives (after simplification)

$$\mathrm{Hom}(\mathbb{Z}_m,\mathsf{SU}(n,1;\Lambda_0))=\left\{A+\lambda X\middle|\begin{matrix}A\in SO(n,1) & JA^{-1}X\in\mathsf{Sym}(n+1;\mathbb{R})\\ A^m=I & X\in\ker\sum_{i=1}^n\mathsf{Ad}_A^i\end{matrix}\right\}$$

For $G\in\{\mathsf{SL}(n+1;\mathbb{R}),\mathsf{SU}(n,1;\mathbb{C})\}$, the tangent space to the representation $\rho:\mathbb{Z}_m\to G$ sending the generator to A is the set of all vectors $X\in T_AG$ which are derivatives of paths $t\to A_t\in G$ with $A_t^m=I$ and $A_0=A$. An immediate computation with the product rule shows X satisfies the relation above. □

An analogous result holds for finitely presented groups: the evaluation map embeds $\mathrm{Hom}(\Gamma,G)$ in G^2, and $T\,\mathrm{Hom}(\Gamma,G)$ in $G^n\times\mathfrak{g}^n$ for $G\in\{\mathsf{SL}(n+1;\mathbb{R}),\mathsf{SU}(n,1;\mathbb{C})\}$. Representations into $\mathsf{SO}(n,1)$ have image in $\mathsf{SO}(n,1)^2$ and the normal bundle to this is $N_G^n\subset G^n\times\mathfrak{g}^n$. Applying the map p_G of observation 6.2 on each generator identifies $\mathrm{Hom}(\Gamma,\mathsf{SU}(n,1;\Lambda_0))$ with $T\,\mathrm{Hom}(\Gamma,G)\cap N_G^n$; infinitesimal paths of representations through the $\mathsf{SO}(n,1;\mathbb{R})$ representations orthogonally.

For triangle groups (and other rigid subgroups of $\mathsf{SO}(n,1)$, such as finite volume hyperbolic manifold groups in dimension ≥ 3) we may simplify things by focusing on a fixed hyperbolic representation $\rho_{\mathbb{H}}\colon\Gamma\to\mathsf{SO}(n,1)$. The homomorphism $\pi\colon\mathsf{SU}(n,1;\Lambda_0)\to\mathsf{SO}(n,1)$ induces a projection $\widetilde{\pi}$ of representation varieties, and we denote $V=\widetilde{\pi}^{-1}(\rho_{\mathbb{H}})$. Rephrasing the above in this restricted context gives the following.

Observation 6.4. The map $p_G^n\colon V\to G^n\times\mathfrak{g}^n$ identifies V with $T_{\rho_{\mathbb{H}}}\mathrm{Hom}(\Gamma,G)\cap N_G^n$. That is, V can be thought of as the tangent vectors to $\rho_{\mathbb{H}}$ in $T_{\rho_{\mathbb{H}}}\mathrm{Hom}(\Gamma,G)$ orthogonal to $T_{\rho_{\mathbb{H}}}\mathrm{Hom}(\Gamma,\mathsf{SO}(n,1))$.

6.3 Deforming Triangle Groups

To set notation, fix a hyperbolic triangle group $\Delta = \Delta(p, q, r)$ with $2 \notin \{p, q, r\}$. Let $\rho \in \mathrm{Hom}(\Delta, \mathsf{SU}(2, 1; \Lambda_0))$ be any representation with real part a discrete and faithful representation $\rho_{\mathbb{H}}$ into $\mathsf{SO}(2, 1)$.

Evaluating on the generators, say $\mathsf{ev}(\rho) = (A + \lambda X, B + \lambda Y)$ and let $\rho_{\mathbb{H}} = \widetilde{\pi}(\rho)$ be the real hyperbolic representation with image generated by A, B. By observation 6.4, $p_{\mathsf{GL}}(\rho) = (\rho_{\mathbb{H}}, \vec{v})$ for $v = (X, Y) \in T_{\rho_H} \mathrm{Hom}(\Delta, \mathsf{GL}(3; \mathbb{R}))$, and similarly $p_{\mathsf{U}}(\rho) = (\rho_{\mathbb{H}}, \vec{w})$ for $\vec{w} = (iX, iY) \in T_{\rho_{\mathbb{H}}} \mathrm{Hom}(\Delta, \mathsf{SU}(2, 1; \mathbb{C}))$. The representation $\rho_{\mathbb{H}}$ is a smooth point of $\mathrm{Hom}(\Delta, \mathsf{SL}(3; \mathbb{R}))$ by [**?**] and so it is also a smooth point of $\mathrm{Hom}(\Delta, \mathsf{SU}(2, 1; \mathbb{C}))$ by [CLT07]. Thus, the tangent vectors $\vec{v}$ and $\vec{w}$ respectively are integrable to paths of representations $f_t \colon (-\varepsilon, \varepsilon) \to \mathrm{Hom}(\Delta, \mathsf{SL}(3; \mathbb{R}))$ and $\phi_t \colon (-\varepsilon, \varepsilon) \to \mathrm{Hom}(\Delta, \mathsf{SU}(2, 1; \mathbb{C}))$ respectively.

As $\mathsf{SU}(2, 1; \mathbb{C}) \cong \mathsf{SU}(2, 1; \Lambda_\delta)$ with some care the path ϕ_t can be lifted to a 1-parameter family of representations $\widetilde{\phi}_t \colon \Delta \to \mathsf{SU}(2, 1; \Lambda_\delta)$ for $t \in (-\varepsilon, 0)$ such that $\lim_{t \to 0^-} \widetilde{\phi}_t = \rho$, and similarly $\widetilde{f}_t$ using $\mathsf{SU}(2, 1; \Lambda_\delta) \cong \mathsf{SL}(3; \mathbb{R})$. We briefly illustrate this for ϕ_t, denote $\mathsf{ev}(\phi_t) = (A_t + iC_t, B_t + iD_t)$ and note that as $t \to 0$, $(A_t, B_t) \to (A, B)$ and $(C, D) \to \vec{0}$ while $(C'_t, D'_t) \to (X, Y)$. The isomorphism $\eta_\delta \colon \mathbb{C} \to \Lambda_\delta$ sending $i \mapsto \lambda/\sqrt{|\delta|}$ acts componentwise on the generators to convert any representation $\phi_t \colon \Delta \to \mathsf{SU}(2, 1; \mathbb{C})$ to to a representation $\phi_{t,\delta} \colon \Delta \to \mathsf{SU}(2, 1; \Lambda_\delta)$ generated by the pair

$$\mathsf{ev}\phi_{t,\delta} = \left(A_t + \frac{\lambda}{\sqrt{|\delta|}} C_t, B_t + \frac{\lambda}{\sqrt{|\delta|}} D_t \right).$$

The limiting behavior as $t, \delta \to 0$ is determined by the relative rate of vanishing; an easy check shows taking $t = -\sqrt{|\delta|}$ achieves the limit $(A + \lambda X, B + \lambda Y)$ as $\delta \to 0^-$. Thus we define $\widetilde{\phi}_\delta = \phi_{-\sqrt{|\delta|}, \delta}$ for $\delta \leq 0$. Similar considerations allow us to lift $\widetilde{f}_t$. To have an explicit example, we consider the triangle group $\Delta = \Delta(3, 3, 4)$. The following path of representations was computed using the explicit description of $\mathsf{SL}(3, \mathbb{R})$ representations from the work of [CLT06].

Example 6.5. Let $\Delta = \langle a, b | a^3 = b^3 = (ab)^4 = 1 \rangle$, and $\rho_u \colon \Delta \to \mathsf{GL}(3, \Lambda_{u^2-8})$ be the following representation.

$$\rho_u(a) = A_u = \begin{pmatrix} 1 & 1 & -(4+u)+\lambda \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix} \qquad \rho_u(b) = B_u = \begin{pmatrix} 0 & 0 & \frac{1}{4}(\lambda - u) \\ \frac{1}{2}(4+u+\lambda) & 1 & -1 \\ \frac{1}{2}(u+\lambda) & 0 & -1 \end{pmatrix}$$

A simple computation checks this is a representation of Δ into $\mathsf{GL}(3, \Lambda_{u^2-8})$ for all u, and thus is complex hyperbolic for $u < 2\sqrt{2}$ and self-dual real projective for $u > 2\sqrt{2}$.

Quotenting the domains of these geometries by the actions of $\widetilde{\phi}_\delta(\Delta)$, $\widetilde{f}_\delta(\Delta)$ gives a geometric realization of this transitional behavior. From the work of [CLT07] it follows that for negative δ close to 0, the quotient $\mathcal{X}_\delta(n, 1)/\widetilde{\phi}_\delta(\Delta)$ is a complex hyperbolic 4-orbifold with fundamental group the triangle group Δ. For $\delta = 0$ the resulting manifold $\mathcal{X}_0(n, 1)/\rho_0(\Delta)$ is a plane bundle over the hyperbolic structure $\mathbb{H}^2/\rho_{\mathbb{H}}(\Delta)$, transitioning to the self-dual real projective picture. For $\delta > 0$, two projections $\mathsf{SU}(2, 1; \Lambda_\delta) \to \mathsf{SL}(3; \mathbb{R})$ given by multiplication by the principal idempotents $e_\pm$ send $\widetilde{f}_\delta$ to the holonomy of a convex real projective structure on $\mathbb{S}^2(p, q, r)$ and its contragredient respectively. The product of these is an open domain in $\mathbb{R}\mathsf{P}^2 \times \mathbb{R}\mathsf{P}^2$ on which $\widetilde{f}_\delta(\Delta)$ acts properly discontinuously; this can then be pulled back to a domain in the unitary geometry for $\mathsf{SU}(2, 1; \Lambda_\delta)$ by its identification with self-dual real projective space.

References

[Bal] Samuel Ballas, *Properly convex bending of hyperbolic manifolds.*

[BDL15] Samuel Ballas, Jeffrey Danciger, and Gye-Seon Lee, *Convex projective structures on non-hyperbolic three-manifolds.*

[Ben06] Yves Benoist, *Convexes divisibles iv : Structure du bord en dimension 3.*

[Ber60] Lipman Bers, *Simultaneous uniformization*, Bull. Amer. Math. Soc. **66** (1960), no. 2, 94–97.

[CDW14] D. Cooper, J. Danciger, and A. Wienhard, *Limits of geometries.*

[CHK00] D. Cooper, C. D. Hodgson, and S. P. Kerckhoff, *Three-dimensional orbifolds and cone-manifolds, volume 5 of msj memoirs*, Mathematical Society of Japan, Tokyo, 2000.

[CLT06] Daryl Cooper, Darren Long, and Morwen Thistlethwaite, *Computing varieties of representations of hyperbolic* 3*-manifolds into slnr*, Experiment. Math. **15** (2006), no. 3, 291–306.

[CLT07] ———, *Flexing closed hyperbolic manifolds*, Geom. Topol. **11** (2007), no. 4, 2413–2440.

[CRPS16] Angel Cano, John R. Parker, and Jose Seade, *Action of* $\mathbb{R}$*-fuchsian groups on* $\mathbb{CP}^2$.

[Dan] J. Danciger, *Ideal triangulations and geometric transitions*, Journal of Topology.

[Dan11] Jeff Danciger, *Geometric transitions: From hyperbolic to ads geometry*, PhD Thesis **Stanford University** (2011).

[Dan13] J. Danciger, *A geometric transition from hyperbolic to anti de sitter geometry*, Geom. Topol. (2013).

[GBPS14] Renato G. Bettiol, Paolo Piccione, and Gaetano Siciliano, *Deforming solutions of geometric variational problems with varying symmetry groups.*

[Hod86] C.D. Hodgson, PhD Thesis (1986).

[HPS01] M. Huesener, J. Porti, and E. Suarez, *Regenerating singular hyperbolic structures from sol*, J. Diff. Geom (2001).

[PLB05] Joan Porti, Bernhard Leeb, and Michel Boileau, *Geometrization of 3-dimensional orbifolds.*

[Por98] J. Porti, *Regenerating hyperbolic and spherical cone structures from euclidean ones*, Topology (1998).

[Por03] Joan Porti, *Regenerating hyperbolic cone structures from nil.*

[R.E] R.E.Schwartz, *Complex hyperbolic triangle groups.*

[RPP10] John R Parker and Ioannis Platis, *Complex hyperbolic quasi-fuchsian groups.*

[RPWX16] John R. Parker, Jieyan Wang, and Baohua Xie, *Complex hyperbolic (3 , 3 ,n) triangle groups*, 433–453.

[Sch01] R.E. Schwartz, *Ideal triangle groups, dented tori, and numerical analysis*, Annals of Math **153** (2001).

[Tre18] Steve Trettel, *Families of geometries and real algebras*, In Preparation (2018).