An Intrinsic Formalism for Transition Geometry

Steve Trettel

Abstract

1 Introduction

Thurston's Geometrization Conjecture [20] placed the study of geometric structures on manifolds at the heart of low dimensional topology. The deformation spaces of such structures are intimately related to representation varieties via the Ehresmann-Thurston principle. As these deformation spaces are often noncompact, it is possible to look at *degenerations* of geometric structures, paths in the deformation space leaving all compact sets, and attempt to understand what sort of geometric data survives in the limit. Sometimes, the correct data to capture this limit involves some new geometric object (such as the measured laminations in the Thurston compactification of Teichmüller space), but other times, the limit can be naturally identified with a geometric structure itself, albeit modeled on a different homogeneous space. A simple example of this is given by hyperbolic conemanifold structures on the three punctured sphere. For any three positive numbers with sum less than 2π, we may give the three punctured sphere a unique locally homogeneous metric of curvature -1, with these three numbers as the cone angles at each cone point. As the sum of these angles approaches 2π, the structures *collapse*: their area tends to zero. However, after appropriately rescaling (increasing curvature towards zero at a rate which keeps the area constant), such a degenerating path of hyperbolic conemanifold structures is seen to converge to a Euclidean metric (the double of a Euclidean triangle) on the thrice punctured sphere.

Such phenomena also occur in higher dimensions and more varied contexts. Hodgson [13] and Porti [17] analyze Euclidean limits resulting from hyperbolic conemanifolds collapsing to a point, which plays an important role in the Orbifold Theorem of Cooper, Hodgson, & Kerckhoff [3] and Boileau, Leeb & Porti [19] generalizing geometrization to certain singular spaces. Further work of Porti studies the nonuniform collapse of hyperbolic structures to Nil [18] and Sol [14]. Collapsing structures may even have non-Riemannian limits, as the transition from hyperbolic to Half Pipe geometry discovered by Danciger [6, 4, 5], and the example of Heisenberg geometry as a limit of 2, $\mathbb{S}^2$ and $\mathbb{E}^2$ [21].

These phenomena motivate the general study of *geometric transitions*, or continuous deformations of a geometry which abruptly change in isomorphism type. Following Klein, a geometry is a space X with automorphism group G. A geometric transition is a continuous path (G_t, X_t) of geometries where the isomorphism type is discontinuous in t. The example that inspires the theory is the continuous family of simply connected model spaces $\mathbb{M}_\kappa$ of constant curvature κ, which are isomorphic to the hyperbolic space for $\kappa < 0$ and the sphere for $\kappa > 0$, transitioning through Euclidean space at $\kappa = 0$, fundamental to the thrice punctured sphere above. Geometric transitions arise naturally in Riemannian geometry and physics. The work of Umehara and Yamada study constant mean curvature tori along the $3 \leftrightarrow \mathbb{S}^3$ transition [22], and Morabito analyzes minimal surfaces along the $\mathbb{H}^2 \times \mathbb{R} \leftrightarrow \mathbb{S}^2 \times \mathbb{R}$ transition [7]. In Lorentzian geometry, transitions give means of realizing the Galilean group as the $c \to \infty$ limit of special relativity [1].

Formalization

As there are many paths which lead one to examples of some sort of geometric transition - like behavior, there have been multiple attemtps to formalize this notion in the literature. On the level of Lie algebras, Inönü-Winger contractions [15] provide a method of formally expressing a transition between groups by varying the structure constants of their Lie algebras. The theory of *conjugacy limits* developed by Cooper, Danciger & Wienhard [2] describes transitional behavior on the level of groups between subgroups of a fixed ambient group G, which has numerous applications to subgeometries of projective geometry [2, 8]. Briefly a collection of subgeometries (G_t, X_t) of $\mathbb{RP}^n$ is *converges* when the groups G_t converge in the Chabauty space of $\mathsf{PGL}(n+1, \mathbb{R})$ and the domains X_t converge appropriately as subsets of $\mathbb{RP}^n$. A subgeometry (H, Y) is a *conjugacy limit* of (G, X) in $\mathbb{RP}^n$ if there is a sequence $\{C_n\} \subset \mathsf{PGL}(n+1; \mathbb{R})$ with $C_n G C_n^{-1} \to H$ and $C_n X \to Y$. These conjugacy limits are the limit points of conjugacy classes in the Chabauty space of G , a topic also studied recently by Haettel [10, 11, 12]. Other Geometric formalizations include the bundle construction of A'Campo & Papadapoulos [16] of "coherent group elements," and certain Lie groupoids studied by Bettiol, Piccione, & Siciliano [9].

1.1 Aims

The goals of this paper are to introduce an intrinsic formalism for geometric transitions, which provides at once a suitable generalization of the formalisms currently in use while also being computationally tractable to work with. More precisely, our goal is to build a theory of geometric transitions which satisfies the following constraints:
Constraint 1: Sufficiently General *Any good definition of a family of geometries must reproduce all current examples in geometric topology: those coming from conjugacy limits, bundles of coherent elements, and bundles of groups.*

Constraint 2: Intrinsic *Any good definition of a family of geometries must be intrinsic; that is, the existence of a transition between geometries X and Y must not be defined relative to a specific chosen embedding of X and Y into some ambient space.*
Constraint 3: Tractable *Any good definition of a family of geometries must be computationally tractable; that is, the theory should provide a means of producing and verifying concrete examples.*

1.2 Results

The results of this paper are summarized below. The main idea for producing a sufficiently general, computationally tractible intrinsic formalism is to take the category Diff of smooth manifolds, and replace it with some category of 'smooth manifolds varying over a base' (so that original manifolds are 'smooth families over a point'). Then with this new category as our starting point, we attempt to replicate the portions of the theory of smooth manifolds needed to develop the definition of homogeneous spaces. We strive to work internally to this new category of families as much as possible; leading us to *define* a families of groups as the group objects in this category, and families of group actions as maps in this category satisfiying the usual diagrams defining an action of Lie groups in Diff. Putting this all together, we define a family of geometries as a *homogeneous space object* in this category.

Definition 1: *A family of geometries parameterized by a smooth manifold B is a triple $(\mathcal{G}, \mathcal{X}, \alpha)$ of families of groups, manifolds $\mathcal{G}, \mathcal{X} \in \mathsf{Fam}_B$ equipped with a surjective action map $\alpha\colon \mathcal{G} \times_B \mathcal{X} \to \mathcal{X} \times_B \mathcal{X}$.*

Develop basic properties of this category, including pullbacks, quotients.

Theorem 1: *Let $\mathcal{G} \curvearrowright \mathcal{X}$ be a proper free action of a family of Lie groups on a family of smooth manifolds parameterized by a base B. Then the family $\mathcal{X} \to B$ factors into a family of families $\mathcal{X} \to \mathcal{X}/\mathcal{G} \to B$ with $\mathcal{X} \to \mathcal{X}/\mathcal{G}$ the smoothly varying family of $\mathcal{G}$-orbits over the orbit space, and $\mathcal{X}/\mathcal{G} \to B$ exhibiting the orbit space as a family over the original base.*

We use this theory to prove that standard things about category of homogeneous spaces carry over: in particular, it is often very convienient in practice to work with a homogeneous space not as a pair of a Lie group G acting transitively on a space X, but rather by remembering only the group G and the closed subgroup $K < G$ stabilizing some point of X (we can then reconstruct X up to diffeomorphism as the coset space G/K). We prove that this perspective is also justified in the category of families.

Theorem 2: *There is an equivalence of categories between the two useful notions of a family of homogeneous spaces: namely that of a family of groups acting transitively on a family of spaces (above) and family of groups together with a family of closed subgroups.*

We next show that our new definition subsumes the previous formalisms of conjugacy limits, and bundles of coherent elements, from the literature

Theorem 3: *Every smooth conjugacy limit of a subgeometry (G, X) of projective space in the sense of [2] s a family of geometries. Likewise, the sets of coherent elements for a* fiber space *transition in the sense of [16] is a family of geometries.*

But, there are more things possible now. A productive means of leveraging this formalism to produce new examples is to attempt to run familiar constructions from the category Diff in the category of families.

Theorem 4: *For each* (p, q) *with* $p, q \neq 0$ *there is a transition between the two nonisomorphic subgeometries of* $\mathbb{R}\mathsf{P}^{p+q-1}$ *with automorphisms* $\mathsf{PO}(p, q)$ *as a family of geometries, but not as a conjugacy limit.*

Corollary 1: *There is a geometric transition of hyperbolic geometry which does not occur via conjugacy limit in* $\mathbb{RP}^n$*. Namely,* $\mathbb{H}^n$ *transitions to de Sitter geometry through the canonical line bundle to conformal geometry on the* $n-1$ *sphere.*

Theorem 5: *For any smooth family of algebras* $\mathcal{A} \to B$ *over a base* B*, there is a corresponding family of projective geometries* $\mathcal{AP}^n \to B$ *with automorphisms given by the family of groups* $SL(n+1; \mathcal{A}) \to B$*.*

Corollary 2: *There is a transition from the geometry of* $\mathbb{C}\mathsf{P}^n$ *to that of* $(\mathbb{R} \oplus \mathbb{R})\mathsf{P}^n \cong \mathbb{R}\mathsf{P}^n \times \mathbb{R}\mathsf{P}^n$*. For* $n = 1$ *this recovers the transition between the ideal boundaries* $\partial_\infty \mathbb{H}^3 = \mathbb{C}\mathsf{P}^1$ *and* $\partial_\infty \mathsf{AdS}^{2,1} = \mathbb{R}\mathsf{P}^1 \times \mathbb{R}\mathsf{P}^1$ *used by Danciger in CITE.*

Theorem 6: *For* $\mathcal{A} \to B$ *a family of real algebras and* J *a real symmetric matrix, there is a subfamily of* $\mathcal{AP}^n$ *of geometries with automorphism groups given by the family of orthogonal groups* $\mathsf{O}(J; \mathcal{A})$*. Similarly if* $\mathcal{A} \to B$ *is equipped with an involution* $\sigma : \mathcal{A} \to \mathcal{A}$*, there is a subfamily of unitary geometries with automorphism groups* $\mathsf{SU}(J, \mathcal{A})$*.*

Applying this to the transition of algebras from $\mathbb{C} \to \mathbb{R} \oplus \mathbb{R}$

Corollary 3: *There is a transition of complex hyperbolic geometry which does not occur as a conjugacy limit inside of complex projective space. Moreover this transition results in an analog of hyperbolic geometry built over* $\mathbb{R} \oplus \mathbb{R}$*, which is a* $2n$ *dimensional homogeneous space for* $\mathsf{SL}(n+1; \mathbb{R})$

Acknowledgements

2 Background

2.1 A Prehistory of Transition Geometry

Bolyai hyperbolic geometry: with a parameter limiting to Euclidean

Definition 2: *(Klein) A geometry is given by the data of a smooth manifold* X*, equipped with a transitive action of a Lie group* G*.*

Geometric properties of the pair (G, X) (such as the existence of a metric, etc) are derived from this.

Klein: construction of this transition inside of Projective geometry

Poincare: Described a 2-d geometry for each quadric in 3-space. This includes limits of orthogonal geometries; discovered de Sitter 1+1 space.

2.2 Collapsing Structures and Transitions

Geometrization

Danciger: new conj limit of Hyperbolic 3-space

2.3 Formalization and Classification

3 Homogeneous Geometry

This section brings together various notions of homogeneous geometry used in geometric topology and the study of symmetric spaces to provide a foundation and set terminology for the coming work. The definitions and results of this section are not original, but to the author's knowledge are not consolidated in any particular place in the geometric topology literature, and so an attempt to be as self-contained and thorough as possible has been made.

Definition 3: *A homogeneous geometry is a triple (G, X, α) of a Lie group G and a smooth manifold X, equipped with an analytic and transitive action $\alpha\colon G \times X \to X$. A morphism of geometries is a pair of smooth maps $(\Phi, F)\colon (G, X, \alpha) \to (H, Y, \beta)$ respecting the actions: that is, Φ is a group homomorphism and for all $g \in G$, $x \in X$ we have $F(\alpha(g, x)) = \beta(\Phi(g), F(x))$.*

We follow Klein and throughout this paper a *Geometry* refers to a homogeneous spade of this form. Following standard practice, we suppress α when this does not overload notation, and write the action of G on X as $\alpha(g, x) = g.x$. Central to the study of homogeneous spaces is the notion of *subgeometries*: indeed Klein's original interest stemmed from rigorously constructing hyperbolic geometry as a subgeometry of projective space.

Definition 4: *A subgeometry of (G, X) is an embedding $\iota\colon (H, Y) \hookrightarrow (G, X)$: a morphism of geometries which is an embedding on each factor. Conflating such a map with its image, a subgeometry is a closed subgroup $H < G$ acting transitively on a subset $Y \subset X$.*

Alternatively in this situation, we say that (G, X) is a *supergometry* or *containing geometry* for (H, Y). Often are interested in particular in *open subgeometries*, where we furthermore require that $Y \subset X$ be open (note, the *subgeometries* of CDW are what we call *open subgeometries* here). An important kind of subgeometry arises by fixing the space X, but restricting the automorphism group from G to some smaller subgroup, making the geometry more rigid in the sense of having less allowable motions.

Definition 5: *Let (G, X) and (H, X) be two geometries with the same underlying manifold X. Then (H, X) is a strictification of (G, X) if there is an embedding $(\Phi, \mathsf{id}_X)\colon (H, X) \hookrightarrow (G, X)$. Conversely, (G, X) is a relaxation of (H, X) in this circumstance.*

This arises in practice when the holonomy of some (G, X) structure on a manifold M actually lies within a transitive subgroup $H < G$ (such as the euclidean group within the affine group). In such cases we may think of the (G, X) structure as an (H, X)

structure when convenient. Somewhat dual to the notion of subgeometry is that of a *fibered geometry.*

Definition 6: *A fibration of* (G, X) *over* (H, Y) *is a morphism* $(G, X) \to (H, Y)$ *which is a submersion on each factor.*

These arise in the disucssion of limits of the orthogonal geometries in CITE, where it is shown that many of the limiting *partial flag geometries* are fibered. A particular case of this, the geometry of the real Heisenberg group acting on the plane fibers over the geometry $(\mathbb{R}, \mathbb{R})$ of the line. The geometric structures on surfaces with holonomy into the real heisenberg group is investigated in CITE, where this fibration plays a useful role in classification.

3.1 Categories of Geometries and Equivalences

When used to model geometric structures on manifolds, the above definition of homogeneous geometries is most convenient. However, when trying to develop the theory of homogeneous geometries themselves, it is often useful to work with a pointed version.

Definition 7: *A pointed geometry* $(G, (X, x), \alpha)$ *is a geometry together with a choice of basepoint* $x \in X$*. Morphisms* $(\Phi, F)\colon (G, (X, x)) \to (H, (Y, y))$ *between pointed geometries respect this by requiring that* $F\colon (X, x) \to (Y, y)$ *be a pointed map.*

Again we drop α from the notation and write $\alpha(g, x) = g.x$ when no ambiguity results. There is a forgetful functor from the category of pointed geometries to the category of geometries defined above simply by forgetting the basepoint, and it is easy to check that given a geometry (G, X) any two choices of basepoint $x_1, x_2 \in X$ yeild isomorphic pointed geometries $(G, (X, x_1)) \cong (G, (X, x_x))$ (though the isomorphism is not cannonical, and the number of possiblities is indexed by $\mathsf{stab}_G(x_1)$. But given a pointed geometry $(G, (X, x))$ there is a canonical choice of stabilizer subgroup $K = \mathsf{stab}_G(x) < G$, and the transitive G action on X provides a diffeomorphism $X \cong G/K$. Thus, an alternative formalism for homogeneous geometry can be built purely Lie-theoretically, forgetting the space X and remembering only the stabilizer of the G action.

Definition 8: *A geometry is a pair* (G, K) *of a Lie group* G *and a closed subgroup* K*. A morphism of geometries is a Lie group homomorphism of pairs* $\Phi\colon (G, K) \to (H, C)$*.*

When necessary to distinguish these two notions, we will call the first the category of pointed geometries and denote it GSpc_*, and the second the category of group/stabilizer geometries and denote it GStb. These two formalisms encode homogeneous geometry in distinct, but equivalent ways.

Lemma 4: *The maps* $(G, (X, x), \alpha) \to (G, \mathsf{stab}_G(x))$ *and* $(G, K) \mapsto (G, (G/K, K), \cdot)$ *are functors between these two categories of geometries.*

Proof. We begin with the assignment $\Psi\colon (G, (X, x), \alpha) \to (G, \mathsf{stab}_G(x))$. First, the stabilizer of an action of a Lie group on a smooth manifold is a closed Lie subgroup, so $(G, \mathsf{stab}_G(x))$ is an object in the correct category. Recalling that a morphism Φ :

$(G,(X,x)) \to (H,(Y,y))$ consists of a group homomorphism Φ_{Grp} and an equivariant map Φ_{Sp} between the spaces, the image $\Psi(\Phi) = \Phi_{\mathsf{Grp}}$ is simply the group homomorphism, which is well-defined as $\Phi_{\mathsf{Sp}}(x) = y$ together with equivariance implies that $\Phi_{\mathsf{Grp}}(\mathsf{stab}_G(x)) \subset \mathsf{stab}_H(y)$. Thus $\Psi(\Phi)$ is a morphism $\Psi((G,(X,x)) \to \Psi(H,(Y,y))$.

Next we consider the second assignment $F\colon (G,K) \to (G,(G/K,K))$ As K is a closed subgroup of G, the K action on G by left translation by is a free and proper action. Thus by the quotient manifold theorem of smooth topology, the orbit space G/K is a smooth manifold. The action of G on G/K is just the usual action of G on itself followed by the quotient map, which is transitive. The inclusion $K \hookrightarrow G/K$ provides a cannonical choice of basepoint. Given a morphism $\Phi : (H,K) \to (G,C)$ we define $\mathsf{F}(\Phi) = (\Phi,\overline{\Phi})$ where $\overline{\Phi}(gC) = \Phi(g)K$. This is Φ-equivariant and well-defined as $\Phi(C) \subset K$, and $\overline{\Phi}(\mathcal{C}) = \mathcal{K}$ as required. ◻

Because a pointed geometry canonically picks out a particular stabilizer subgroup of the G action on X (and likewise a chosen closed subgroup K picks out canonically a point of the coset space G/K), these are an equivalence of categories.

Proposition 5: *The functors* F*,* Φ *above define an equivalence of categories* $\mathsf{GSpc}_* \simeq \mathsf{GStb}$.

Proof. The composition $\Psi\mathsf{F}$ is the identity on GrpStb, and the composition $F\Psi$ takes the geometry $(G,(X,x))$ to $(G,(G/\mathsf{stab}_G(x)),\mathsf{stab}_G(x))$ in GrpSpc_*. To complete the proof we show the collection of maps $\eta|_{(G,X)}\colon (G,(X,x)) \to (G,(G/\mathsf{stab}_G(x),\mathsf{stab}_G(x)))$ given below forms a natural transformation from $\mathsf{id}_{\mathsf{GrpSp}}$ to $\mathsf{F}\Psi$.

In more detail, η is given by $\eta = (\mathsf{id}_G, \xi_{(G,X)})$ where $\xi_{(G,X)}$ assigns to a point $p \in X$ the coset $g\mathsf{stab}_G(x)$ of the basepoint stabilizer, for g such that $\mathsf{stab}_G(p) = g\mathsf{stab}_G(x)g^{-1}$. To see this it suffices to check that $\overline{\Phi_{\mathsf{Grp}}} \circ \xi_{(G,X)} = \xi_{(H,Y)} \circ \Phi_{\mathsf{Sp}}$. Let $p \in X$ and $g \in G$ be such that $g.x = p$. Then $\xi_{(G,X)}(p) = g\mathsf{stab}_G(x)$ and $\overline{\Phi_{\mathsf{Grp}}}(g\mathsf{stab}_G(x)) = \Phi_{\mathsf{Grp}}(g)\mathsf{stab}_H(y))$. Computing the other way around we find $\Phi_{\mathsf{Sp}}(p) = \Phi_{\mathsf{Sp}}(g.x) = \Phi_{\mathsf{Grp}}(g)\Phi_{\mathsf{Sp}}(x) = \Phi_{\mathsf{Grp}}(g)y$ and $\xi_{(H,Y)}(\Phi_{\mathsf{Grp}}(g)y)) = \Phi_{\mathsf{Grp}}(g)\mathsf{stab}_H(y)$.

$$\begin{array}{ccc}
(G,(X,x)) & \xrightarrow{(\mathsf{id}_G,\xi_{(G,X)})} & (G,(G/\mathsf{stab}_G(x),\mathsf{stab}_G(x))) \\
{\scriptstyle(\Phi_{\mathsf{Grp}},\Phi_{\mathsf{Sp}})}\downarrow & & \downarrow{\scriptstyle(\Phi_{\mathsf{Grp}},\overline{\Phi_{\mathsf{Grp}}})} \\
(H,(Y,y)) & \xrightarrow[(\mathsf{id}_H,\xi_{(H,Y)})]{} & (H,(H/\mathsf{stab}_H(y),\mathsf{stab}_H(y)))
\end{array}$$

◻

This equivalene of categories is often used implicitly, allowing one to define a particular property of homogeneous geometry in whichever formalism is more convenient. Where the distinction on which particular category is being considered is not relevant to the given discussion, we will denote by Geo any of the above categories of homogeneous geometries.

Effective Equivalence & Local Isomorphism

Homogeneous geometries provide model spaces for geometric structures, and when viewed with this application in mind it is clear that many formally distinct geometries (in any of the three definitions above) are equivalent from the perspective of geometric structures. Examples of this include whether the automorphisms of $\mathbb{H}^2$ are thought of as $\mathsf{SL}(2,\mathbb{R}) \cong \mathsf{SO}_0(2,1)$, or $\mathsf{SO}(2,1)$ or $\mathsf{PO}(2,1)$, or if we consider the atlas of charts for a spherical structure to take values in $\mathbb{S}^2$ or in $\mathbb{R}\mathsf{P}^2$ equipped with the round metric. To formalize this, we need a weaker notion than isomorphism in Geo, called *equivalence* and developed in two stages below.

Definition 9: *A geometry (G,X) is effective if $g.x = x$ for all $x \in X$ implies that $g = e$. That is, representation $G \to \mathsf{Diffeo}(X)$ induced by the action is faithful. The full subcategory of effective geometries is denoted $\mathsf{Eff} \subset \mathsf{Geo}$.*

Definition 10: *Two geometries (G,X) and (H,X) are effectively equivalent if the actions of G,H on X induce homomorphisms $G,H \to \mathsf{Diffeo}(X)$ with the same image.*

For an example, the effective model of real projective geometry in dimension n is given by the tuple $(\mathsf{PGL}(n+1;\mathbb{R}),\mathbb{R}\mathsf{P}^n)$; however it is effectively equivalent to consider the geometries given by $\mathbb{R}\mathsf{P}^n$ with automorphism group $\mathsf{SL}(n+1;\mathbb{R})$ or even $\mathsf{GL}(n+1;\mathbb{R})$. Given any geometry (G,X), we may construct an effectively equivalent version by dividing out G by the subgroup of transformations acting trivially.

Lemma 6: *The effectivization map defines a functor from Geo to the full subcategory of effective geometries.*

Proof. Let $F\colon (G,X) \to (H,Y)$ be a morphism of geometries. Choosing basepoints $x, y = F_{\mathsf{sp}}(x)$ we work in GrpSp_*, or equivalently in GrpStb, where we may take F to be a Lie group homomorphism of pairs $F\colon (G,K) \to (H,C)$. In this language effectivization divides out a geometry (G,K) by the *normal core* $\mathsf{c}_G(K)$ of K in G - the intersection of all conjugates of K. But as $F(K) < C$ by assumption, $F(\mathsf{c}_G(K)) \subset \mathsf{c}_H(C)$ so F descends to the quotient $\overline{F}\colon (G/\mathsf{c}_G(K), K/\mathsf{c}_G(K)) \to (H/\mathsf{c}_H(C), C/\mathsf{c}_H(C))$ which we define to be $\mathsf{Eff}(F)$. □

Effective equivalence counts two geometries as the same when their effectivizations are isomorphic. This is a condition about the automorphism groups of the geometries, but to finish formalizing the type of equivalence used in practice, we need also a condition on the underlying spaces. The correct notion here is a *local isometry*, as described for instance in CDW.

Definition 11: *A local map $X \dashrightarrow Y$ is a map from some open set $U \subset X$ into Y. A local map of pointed spaces further preserves basepoints. A local homomorphism $\phi\colon (G,e) \dashrightarrow (H,e)$ is a local pointed map such that $\phi(gh) = \phi(g)\phi(h)$ and $\phi(g^{-1}) = \phi(g)^{-1}$ whenever all terms are defined.*

Because the maps in question are require to be local, we can phrase everyting in the Group-Stabilizer formalism purely in terms of Lie algebras.

Definition 12: *A local morphism of geometries* $(G, K) \dashrightarrow (H, C)$ *is a homomorphism of pairs of their Lie algebras* $\phi\colon (\mathfrak{lie}(G), \mathfrak{lie}(K)) \to (\mathfrak{lie}(H), \mathfrak{lie}(C))$. *Two geometries* $(G, K) \simeq (H, C)$ *are locally isomorphic if there is an isomorphism of Lie algebras* $\phi\colon \mathfrak{lie}(G) \to \mathfrak{lie}(H)$ *carrying* $\mathfrak{lie}(K)$ *to* $\mathfrak{lie}(C)$.

The notion of equivalence used in practice is the combination of these two above.

Definition 13: *Two geometries* (G, X) *and* (H, Y) *are equivalent if their effectivizations are locally isomorphic.*

Rather than trying to construct a category of homogeneous spaces by quotienting by this equivalence relation (where we would have to deal with the question of what should be done about the morphisms) it is much easier to work with the categories already introduced and freely switch between equivalent representations when convenient. Indeed, this is what is already done in practice.

4 Families of Smooth Manifolds

With the end goal of developing a theory of continuously varying homogeneous geometry to mimic as much as possible the standard theory recalled in the previous section, we start by laying out the plan. First, we fix a definition for what a smoothly varying family of smooth manifolds parameterized by some base space B should be, and then make note of some basic properties of the category Fam_B of these objects. Replacing the smooth category with Fam_B, we develop an internal theory of homogeneous spaces by constructing families of groups as the groups internal to Fam_B (in direct analogy to how Lie groups are the groups internal to Diff), an internal theory of family-of-group actions, homomorphisms of families of groups, and subfamilies of groups. This provides the necessary definitional scaffolding to mimic the three definitions of homogeneous spaces from Section 3. To prove the analogous equivalence of categories and correctly define effective equivalence and local isometry however, we need also to develop analogs of a couple more tools from smooth topology in the category Fam_B.

This all begins with a definition of a family of manifolds. Geometrically, such a thing should be like a fiber bundle, with a total space, and a parameterizing base, without the restriction of local triviality (with the goal of modeling geometric transitions, we should keep in mind that even in the most familiar example, the $\mathbb{S}^n$ to $\mathbb{H}^n$ transition, the fiber changes homotopy type.

Constraint 1: *Any good definition of a family of geometries must reproduce all current examples: MAKE MORE SPECIFIC*

To satisfy this constraint, neither fiber bundles nor fibrations in the category of smooth manifolds suffice. Instead, the right generalization for our purposes is to relax the condition that the projection $\pi\colon E \to B$ in a fiber bundle be proper. The result is often called a smooth fibered manifold/ smooth family of manifolds, defined below.

Definition 14: *A smooth family of manifolds is a triple* $(\mathcal{X}, B, \pi)$ *consisting of smooth manifolds* $\mathcal{X}, B$ *equipped with a smooth submersion* $\pi\colon \mathcal{X} \to B$. *The space* $\mathcal{X}$ *is the* total space*, B the* base*, and the fibers $\mathcal{X}_b := \pi^{-1}\{b\}$ are the* members *of the family.*

A family contains a transition if there are nonisomorphic members over a single connected component of the base. A smooth manifold X is said to *have transitions* if it is a member in some transitioning family. Otherwise, X is said to be *rigid*. The members of $\mathcal{X} \to B$ vary continuously over B *by definition* - we will see later (CITE) that a family $\pi\colon \mathcal{X} \to B$ induces a continuous map $B \to \mathsf{Closed}(\mathcal{X})$ sending $b \mapsto X_b$ so we may think of continuity in this sense as well.

Definition 15: *A morphism between two families $(\mathcal{X}, B, \pi)$ and $(\mathcal{X}', B', \pi')$ is a pair of smooth maps $(\mathcal{F}, f)\colon (\mathcal{X}, B) \to (\mathcal{X}', B')$ commuting with the projections π, π'. The resulting category of families of smooth manifolds is denoted* Fam.

When discussing a morphism $(\mathcal{F}, f)$ between families, the map $\mathcal{F}$ between total spaces is said to *lie over* the map $f\colon B \to B'$ between the bases. Except in a few circumstances (e.g. the construction of pullbacks) we will not concern ourselves with the entirety of Fam but rather focus on subcategories determined by fixing a particular base B.

4.1 The Category Fam_B

Fix a base manifold B (for many examples it is enough to consider the real line), we study the category of families over B:

Definition 16: *For each smooth manifold B, Fam_B is the subcategory of Fam containing all families with base B and morphisms lying over the identity on B.*

That is, morphisms of families $(\mathcal{X}, B, \pi_{\mathcal{X}}) \to (\mathcal{Y}, B, \pi_{\mathcal{Y}})$ are smooth maps $\mathcal{F}\colon \mathcal{X} \to \mathcal{X}'$ forming a commuting triangle with the projections: $\pi_{\mathcal{X}} = \mathcal{F}\pi_{\mathcal{Y}}$. For each $b \in B$, such a morphism $\mathcal{F}$ maps the member $\mathcal{X}_b$ above b in $\mathcal{X}$ to the corresponding member $\mathcal{Y}_b \subset \mathcal{Y}$. We denote this restriction by $\mathcal{F}|_b\colon \mathcal{X}_b \to \mathcal{Y}_b$. When there is no ambiguity, families $\mathcal{X} \xrightarrow{\pi} B$ in Fam_B will be denoted only by their total space $\mathcal{X}$.

It's an easy exercise to see that for $B = \{*\}$ a point, the category Fam_* is isomorphic to the category of smooth manifolds. Thus for each B, we may think of Fam_B as a generalization of the category of smooth manifolds, and develop some of the analogous constructions.

The initial object of Fam_B is the empty family $\varnothing \to B$, generalizing the empty manifold, and the generalization of a point is given by the terminal object: the trivial family $\mathsf{id}_B\colon B \to B$. As a test case of generalizing definitions from Diff to Fam_B, we use these facts to consider sumbersions. A submersion in the differentiable category is a smooth map which is surjective on all tangent spaces: equivalently a map $f\colon X \to Y$ is a submersion if it is transverse to all maps $\{*\} \to Y$ from a point (the terminal object). Generalizing to Fam_B, we say a map $\mathcal{F}\colon \mathcal{X} \to \mathcal{Y}$ is a *submersion of families* if it is transverse to all maps of the terminal object $B \xrightarrow{\mathsf{id}} B$ into $\mathcal{Y} \to B$. Unpacking this, we see that $\mathcal{F}$ is a submersion of families if for each $b \in B$ the restriction $\mathcal{F}|_b\colon \mathcal{X}_b \to \mathcal{Y}_b$ is a submersion.

Definition 17: *A submersion of families $\mathcal{F}\colon \mathcal{X} \to \mathcal{Y}$ is a smooth morphism of families which is memberwise a submersion.*

As a general rule of thumb, the correct Fam_B-generalization of a class of maps having property P in the smooth category, is the class of morphisms which restrict memberwise to have P. This is again true for the next example; monomorphisms in Fam_B are the injective morphisms $\mathcal{F}\colon \mathcal{X} \to \mathcal{Y}$, which restrict memberwise to injective immersions $\mathcal{X}_b \hookrightarrow \mathcal{Y}_b$.

Definition 18: *An embedding of $\mathcal{Z}$ into $\mathcal{X}$ is an injective immersion $\mathcal{Z} \hookrightarrow \mathcal{X}$ which is a diffeomorphism onto its image. A subfamily of $\mathcal{X}$ is the image of an embedding $\mathcal{Z} \hookrightarrow \mathcal{X}$ in Fam_B.*

The product of two families $\mathcal{X}, \mathcal{Y}$ in Fam_B is given by the fiber product $\mathcal{X} \times_B \mathcal{Y}$, equipped with the projection $\mathcal{X} \times_B \mathcal{Y} \to B$ coming from composing two edges of the defining pullback square. The coproduct of two families is the disjoint union of their total spaces, projection maps. Fam_B has all finite products, coproducts. One final construction that will be of much importance later on; pullbacks.

Definition 19: *Let $\mathcal{X} \to B$ be a family, and $f\colon D \to B$ a smooth map. Then the pullback family $f^\star\mathcal{X} \to D$ has total space $\mathcal{X} \times_B D = \{(x,d) \mid f(d) = \pi(x)\}$ and projection map $f^\star\mathcal{X} \xrightarrow{\pi^\star} D$ defined by $(x,d) \mapsto d$.*

Lemma 7: *The pullback family $f^\star\mathcal{X} \to D$ of a family $\mathcal{X} \to B$ exists along any smooth map $f\colon D \to B$.*

Proof. First, we see that the total space $f^\star\mathcal{X}$ is a smooth manifold as it is the pullback in Diff of a smooth manifold along a submersion. It remains to show that the projection map $f^\star\mathcal{X} \xrightarrow{f^\star\pi} D$ defined by $(x,d) \mapsto d$ is a submersion. We do so by recalling that an equivalent property to being a submerison in the smooth category is admitting local sections through each point in the domain.

Let $(x,d)\,\mathrm{in}f^\star\,\mathcal{X} = \mathcal{X} \times_B D$. Then $b = f(d) = \pi(x)$ so $x \in X_b$. As $\mathcal{X} \to B$ is a submersion let $\sigma : V \to \mathcal{X}$ be a local section of π through x. Pulling back gives a map $\sigma \circ f : f^{-1}\{V\} \to \mathcal{X}$ from which the map $f^\star\sigma = (\sigma f, \mathsf{id}_D) : f^{-1}\{V\} \to \mathcal{X} \times D$ can be created. As $\pi(\sigma(f(d))) = f(d)$, the map F has image in $\mathcal{X} \times_\Delta D$, and $\pi^\star \circ (f^\star\sigma)(d) = \pi^\star(\sigma(f(d)), d) = d$ so $f^\star\sigma$ is a section. Finally noting $f^\star(d) = (\sigma(f(d)), d) = (\sigma(\delta), d) = (x,d)$ shows $f^\star\sigma$ is a section of $\pi^\star$ through (x,d). ◻

Lemma 8: *A morphism $D \xrightarrow{f} \Delta$ induces a functor $\mathsf{Fam}_\Delta \xrightarrow{f^\star} \mathsf{Fam}_D$.*

Proof. If $\Phi\colon \mathcal{X} \to \mathcal{Y}$ is a morphism of families over Δ and $f \in \mathrm{Hom}(D, \Delta)$ then $f^\star\Phi\colon f^\star\mathcal{X} \to f^\star\mathcal{Y}$ defined by $(f^\star\Phi)(x,d) = (\Phi(x), d)$ is a morphism of families. This assignment obviously respects composition, as $f^\star(\Phi \circ \Psi) = (f^\star\Phi) \circ (f^\star\Psi)$ and so the operation of pullback defines a functor $\mathsf{Fam}_\Delta \to \mathsf{Fam}_D$ ◻

A trivial consequence of this: we can pull back the smooth manifold category into any category of families over any base B along the constant map $B \to *$. The result?

Each smooth manifold M pulls back to the trivial family of M over B. Pullbacks will be used in the final section on applications in the context of a geometric way of solving equations between families of groups.

Observation 1: Let $\mathcal{X}, \mathcal{Y}$ be objects in Fam_B and $\Phi\colon \mathcal{X} \to \mathcal{Y}$ a morphism of families which is a smooth submersion. Then the pullback $\sigma^\star\mathcal{X}$ of $\mathcal{X} \to \mathcal{Y}$ along any section $\sigma\colon B \to \mathcal{Y}$ is a subfamily of $\mathcal{X} \to B$.

Proof. The pullback $\sigma^\star\mathcal{X} \to B$ is a family over B, and so it suffices to show that the projection map $\mathsf{pr}\colon \mathcal{X} \times B \to \mathcal{X}$, $(x, u) \mapsto x$ is an embedding on $\sigma^\star\mathcal{X}$. But if $\mathsf{pr}(x, u) = \mathsf{pr}(y, v)$ then $x = y$ so $\sigma(u) = \Phi(x) = \Phi(y) = \sigma(v)$ and hence $u = v$ as σ is injective. ◻

As seen in Observation 1, given an equation $\mathcal{X} \xrightarrow{\Phi} \mathcal{Y}$ between certain families over B, the pullback along any section of $\mathcal{Y} \to \Delta$ captures the solutions to $\Phi = \sigma$ as a subfamily of $\mathcal{X}$. Many natural objects can be defined as the solution sets to such equations (point stabilizers are $\{g \mid g.x = x\}$, orthogonal groups are $\left\{A \mid A^T JA = J\right\}$ etc) and so understanding when a map $\Phi \in \mathrm{Hom}_{\mathsf{Fam}_\Delta}(\mathcal{X}, \mathcal{Y})$ actually gives a family $\Phi\colon \mathcal{X} \to \mathcal{Y}$ will be of substantial use.

4.2 Families of Algebraic Gadgets

We now move to defining families of groups, rings, Lie algebras, and other algebraic gadgets. We begin with a straightforward definition for a family of groups capturing what we expect of a geometric transition: a collection of groups where both their underlying manifolds, as well as the group operation, vary continuously.

Definition 20: *A family of groups over a smooth base B is a family $\mathcal{G} \to B$ together with a global section $e\colon B \to \mathcal{G}$ and maps $\mu\colon \mathcal{G} \times_B \mathcal{G} \to \mathcal{G}$, $\iota\colon \mathcal{G} \to \mathcal{G}$ such that each member G_b has the structure of a group with multiplication $\mu|_b\colon G_b \times G_b \to G_b$, inversion $\iota|_b\colon G_b \to G_b$ and identity $e(b)$.*

In practice we surpress the map μ writing groups multiplicatively, and write $\iota(g) = g^{-1}$. By this definition, a family of groups is actually a special type of Lie groupoid $\mathcal{G} \rightrightarrows B$ with the source map $s\colon \mathcal{G} \to B$ and target $t\colon \mathcal{G} \to B$ equal. These Lie groupoids are also known as *Bundles of Groups* in the literature, for example CITE where they are used to model some transitional behavior in Riemannian geometry.

We now turn to rephrase this definition more categorically, to motivate the definition schema for constructing other algebraic gadgets. If we look at what it means to be a group on each fiber, we see that the maps μ, ι, e must satisfy the following commutative diagrams.

$$\begin{array}{ccccc}
(\mathcal{G} \times_B \mathcal{G}) \times_B \mathcal{G} & & \xrightarrow{\quad\cong\quad} & & \mathcal{G} \times_B (\mathcal{G} \times_B \mathcal{G}) \\
\downarrow{\scriptstyle \mu \times_B \mathrm{id}_{\mathcal{G}}} & & & & \downarrow{\scriptstyle \mathrm{id}_{\mathcal{G}} \times_B \mu} \\
\mathcal{G} \times_B \mathcal{G} & \xrightarrow{\ \mu\ } & \mathcal{G} & \xleftarrow{\ \mu\ } & \mathcal{G} \times_B \mathcal{G}
\end{array}$$

$$
\begin{array}{ccc}
\mathcal{G}\times_B B & \xrightarrow{(e,1_{\mathcal{G}})} & \mathcal{G}\times_B \mathcal{G} \\
{\scriptstyle (1_{\mathcal{G}},e)}\downarrow & \searrow^{\equiv} & \downarrow{\scriptstyle \mu} \\
\mathcal{G}\times_B \mathcal{G} & \xrightarrow{\mu} & \mathcal{G}
\end{array}
\qquad
\begin{array}{ccccc}
\mathcal{G}\times_B \mathcal{G} & \xleftarrow{\Delta} & \mathcal{G} & \xrightarrow{\Delta} & \mathcal{G}\times_B \mathcal{G} \\
 & & \downarrow{\scriptstyle \pi} & & \\
{\scriptstyle 1_{\mathcal{G}}\times\iota}\downarrow & & B & & \downarrow{\scriptstyle \iota\times 1_{\mathcal{G}}} \\
 & & \downarrow{\scriptstyle e} & & \\
\mathcal{G}\times_B \mathcal{G} & \xrightarrow{\mu} & \mathcal{G} & \xleftarrow{\mu} & \mathcal{G}\times_B \mathcal{G}
\end{array}
$$

Replacing the product in Fam_B with the usual cartesian product and family morphisms with set functions, these diagrams are exactly the defining axioms of a group. Thus, we may interpret the above definition of a family of groups as *what you get if you try to build a group, but are not allowed to look outside of the category* Fam_B. The result is a *group internal to* Fam_B or a *group object of* Fam_B, giving an equivalent definition. CITE ATWOOD BOOK CH4 GROUPS AND CATEGORIES

Definition 21: *A family of groups is a group object in the category* Fam_B.

This is exactly analogous to familiar constructions, where topological groups being the group objects of Top and Lie groups the group objects of Diff. And this provides a template for moving forwards: to try and develop a theory of homogeneous geometry that naturally includes geometric transtions, we will build the *internal theory of homogeneous geometry* in the category Fam_B, constructing families of rings, fields, vector spaces, and Lie algebras internally to Fam_B along the way. Before carrying on however, one result for topological groups (if a subset of a topological group has a dense subgroup, then it is a subgroup) has a useful generalization here is what follows: if a family is almost a family of groups, then it is a family of groups.

Lemma 9: *Let $\mathcal{G}\to\Delta$ be a family of groups and $\mathcal{H}\to\Delta$ a subfamily of spaces. Then if $\Omega\subset\Delta$ is a dense open subset and $\mathcal{H}|_\Omega$ is a family of groups, all members of $\mathcal{H}$ are groups.*

Proof. Let $\delta\in\partial\Omega$ and $x,y\in\mathcal{H}_\delta$. Choosing sections σ_x, σ_y through them, we may their product $\sigma_x\cdot\sigma_y$ is well defined in $\mathcal{G}$ and lies in $\mathcal{H}$ on the open dense subset $\mathcal{H}|_\Omega$. But $\mathcal{H}$ is closed so in fact the image of $\sigma_x\cdot\sigma_y$ lies fully in $\mathcal{H}$. In particular, $(\sigma_x\cdot\sigma_y)(\delta)=xy$ so $xy\in\mathcal{H}_\delta$. Similarly, as inversion is continuous on $\mathcal{G}$ the section $(\sigma_x)^{-1}$ has image in $\mathcal{H}$ so $x^{-1}\in\mathcal{H}_\delta$. Thus $\mathcal{H}_\delta$ has the structure of a group. □

To work with families of groups, we require homomorphisms of families, which we define internally: a homomorphism of families is a solution in Fam_B to the usual commutative diagrams in Set determining a homomorphism.

Definition 22: *A homomorphism of families of groups is a morphism $\Phi\colon\mathcal{G}\to\mathcal{H}$ making the following diagrams commute:*

$$
\begin{array}{ccc}
\mathcal{G}\times_B \mathcal{G} & \xrightarrow{\Phi\times\Phi} & \mathcal{H}\times_B \mathcal{H} \\
{\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\
\mathcal{G} & \xrightarrow{\Phi} & \mathcal{H}
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{\Phi} & \mathcal{H} \\
{\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle \iota} \\
\mathcal{G} & \xrightarrow{\Phi} & \mathcal{H}
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{\Phi} & \mathcal{H} \\
{\scriptstyle e}\uparrow & \nearrow_{e} & \\
B & &
\end{array}
$$

This immediately implies they are homomorphisms on each slice, by pulling back along the inclusion $* \mapsto b \in B$ for each b. A third useful way of thinking about family homomorphisms is in terms of local sections, which capture local behavior not at a point but in a small neighborhood. The natural definition of homomorphism has an alternative, phrasing in terms of local sections: a map $\Phi\colon \mathcal{G} \to \mathcal{H}$ is a homomorphism of families of groups if for all local sections $\sigma, \tau\colon U \to \mathcal{G}$, we have $\sigma(-)\tau(-)^{-1} = \Phi(\sigma(-))\Phi(\tau(-))^{-1}$ as sections $U \to \mathcal{H}$. Having defined morphisms, we may talk about the subcategory of Fam_B containing families of groups - which will be referred to as Grp_B. To finish, we remark a few basic properties of this category that are useful in the future.

Proposition 10: *The category Grp_B of families of groups has a zero object and finite products but not coproducts.*

Proof. The final object for a family of groups, inherited from the category of families itself, is the trivial family $B \to B$ (thought of as the trivial group at each point of the base). This is also initial as a family of groups, and so is the zero object. Products are simply the products from Fam_B, together with the appropriate products of the multiplication and inversion maps. To see there is no coproduct, it suffices to observe this in Fam_*, or the category of smooth manifolds, where there is already no coproduct of Lie groups. □

Recalling that families of groups are Lie groupoids with equal source and target, Grp_B is an especially nice category of Lie groupoids, and instead of further developing the material here we point the interesrted reader to CITE.

Following this template, we similarly define families of rings and modules internally to Fam_B.

Definition 23: *A family of rings is a ring-object in Fam_B. That is, a family of abelian groups $\mathcal{R} \to B$ together with a multiplication $\mu\colon \mathcal{R} \times_B \mathcal{R} \to \mathcal{R}$ and sections 0, $1\colon B \to \mathcal{R}$ giving each member the structure of a ring.*

Definition 24: *A family of modules over a family of rings $\mathcal{R}$ is a family of abelian groups $\mathcal{M}$ together with an action map $\mathcal{R} \times_B \mathcal{M} \to \mathcal{M}$ giving each member M_b the structure of an R_b-module.*

A family of fields is a family of rings where each member is a field. Families of modules over families of fields are families of vector spaces. Unsurprisingly, many of the standard constructions from algebra carry over internally (products, tensor products etc).

Definition 25: *A family of algebras is a family of vector spaces $\mathcal{A} \to B$ over a family of fields $\mathcal{F} \to B$ equipped with a bilinear multiplication $\mu\colon \mathcal{A} \times_B \mathcal{A} \to \mathcal{A}$ giving each member A_b the structure of an F_b algebra.*

Definition 26: *A family of Lie algebras is a family of vector spaces $g \to B$ over a family of fields, equipped with an alternating bilinear map $[,]\colon g \times_B g \to g$ giving each member the structure of a Lie algebra.*

The categories of manifolds and groups do not have many examples of rigid objects; it is relatively easy to construct nontrivial families containing your favorite group or manifold. However with more algebraic structure this becomes more difficult, and fields, vector spaces are examples of rigid objects: there's no transitions in any family of these. Moreover, families of vector spaces aren't just algebraically trivial, they're topologically trivial as well giving the following.

Proposition 11: *A family of vector spaces is a vector bundle.*

Proof. Proof is straightforward: underlying family of fields is constant (the only locally compact connected fields are $\mathbb{R}$ and $\mathbb{C}$, a simple dimension count shows every family of fields then must have constant isomorphism type), and the dimension of members of a family is invariant; so each member vector space is of the same dimension over the same field. In fact more is true: any family of vector space is also topologically trivial: choosing a basis for any member V_b we may take local sections of the projection $\mathcal{V} \to B$ through each basis element, and use this to construct a local trivialization on a sufficiently small neighbhorhood. For details see THESIS. □

This gives a simpler, equivalent characterization for families of algebras and Lie algebras that is useful in practice.

Corollary 12: *A family of algebras is a vector bundle together with a smoothly varying choice of multiplication on the fibers. A family of Lie algebras is a vector bundle with smoothly varying choice of bracket.*

And actually recovers what is usually meant by a family of algebras CITE, and shows that a family of Lie algebras is the object also known as a weak Lie algebra bundle in the literature CITE. Restricting further to families over a contractible base such as $\mathbb{R}$, we may equivalently think of an algebra as a fixed vector space V together with a 1-parameter family of multiplications, and a Lie algebra as a vector space equipped with a 1-parameter family of brackets.

Proposition 13: *Every family of groups has a family of Lie algebras, but the converse is not true.*

Proof. Let $\mathcal{G}$ be a family of groups in Fam_Δ. Then $\mathcal{G}$ is a smooth manifold with tangent bundle $T\mathcal{G}$. The family projection $\pi\colon \mathcal{G} \to \Delta$ is a smooth submersion, defining the sub-bundle $T^\pi\mathcal{G} = \bigcup_{\delta\in\Delta} T\pi^{-1}(\delta) \subset T\mathcal{G}$ consisting of the tangent bundles to each $\mathcal{G}_\delta$. The tangent spaces at the identity e_δ of each fiber $\mathcal{G}_\delta$ form the pullback bundle $g := e^*(T^\pi\mathcal{G}) \to \Delta$, which each inherit a natural Lie algebra structure arising from $\mathcal{G}_\delta$. Thus it only remains to show that these Lie algebra structures vary continuously over Δ.

Let $\delta \in \Delta$ and $v, w \in \mathfrak{g}_\delta$. Then let $\sigma, \tau\colon U \to g$ be sections of $g \to \Delta$ through v, w respectively. Define the vector fields V, W as the left-invariant vector fields generated by σ, τ: for any $g \in \mathcal{G}_t \subset \mathcal{G}|_U$, $V(g)$ is equal to the pushforward of $\sigma(t)$ by the derivative of the homeomorphism induced by some section α of $g \to \Delta$ through g and similarly for W. Then $[V, W]$ is the vector field defined by $[V, W](f) = V(W(f)) -$

$W(V(f))$ for $f \in C^\infty(\mathcal{G}|_U)$ and $[v,w] = [V,W]_p$, so the Lie bracket structure arises from a continuous construction on vector fields of $\mathcal{G}$.

The fallacy of the converse follows from CITE, where it is shown that every weak Lie algebra bundle is integrable in some sense (there is a corresponding generalized Lie groupoid) but that this is not always Hausdorff - and so certainly is not a family of groups. □

Even more delicate is what happens with subfamilies: Lie subalgebras of $\mathfrak{g}$ are closed subset of Grassmanian, taking any path of these gives a weak Lie algebra bundle, but often exponentiating does not give even necessarily give a continuous path of groups!

Example 1: Let $G = \mathbb{S}^1 \times \mathbb{R}$ and consider the trivial family $G \times \mathbb{R} \to \mathbb{R}$ with corresponding trivial abelian Lie algebra family $\mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$. Let $\mathfrak{h}_t \leqslant \mathbb{R}^2$ be the one dimensional Lie algebra $\mathfrak{h}_t = \mathbb{R}(\cos t, \sin t)$ with exponential $H_t = \left\{(e^{is\cos t}, s\sin t) \mid s \in \mathbb{R}\right\}$. Then the collection $\mathcal{H} = \bigcup_{t\in\mathbb{R}} H_t \times \{t\}$ is *not* a subfamily of $G \times \mathbb{R}$, as the groups H_t are not even Chabauty continuous in G. Indeed as $t \to 0$ the geometric limit of the H_t is the entire cylinder $\mathbb{S}^1 \times \mathbb{R}$, but the group $H_t = \left\{(e^{is}, 0) \mid s \in \mathbb{R}\right\}$ is just the $\mathbb{S}^1$ factor.

This is a computational difficulty for studying conjugacy limits and Chabauty compactifications. For partial results see CITE THESIS. Here we concern ourselves with a more specific lemma, which will be useful to us in connecting the theory of families with conjugacy limits.

Proposition 14: *Let $\mathcal{H} \subset \mathcal{G}$ be a closed submanifold such that each fiber $\mathcal{H}_\delta$ is a group, and the Lie algebras $h \to \Delta$ form a subfamily of $g \to \Delta$. If each $\mathcal{H}_\delta$ is connected, then $\mathcal{H}$ is a subfamily of $\mathcal{G}$.*

If some $\mathcal{H}_\delta$ are disconnected, then under the additional assumption at least one point of each connected component is contained in the image of a section $\sigma\colon U \to \mathcal{H}$, the collection $\mathcal{H}$ is also a subfamily of $\mathcal{G}$.

Proof. Let $A \in \langle exp(h)\rangle$ with $\pi(A) = \delta$. Then $A = A_1 \cdots A_n$ for $A_i \in \exp(\mathfrak{h}_\delta)$, and so $A_i = \exp(X_i)$ for some $X_i \in \mathfrak{h}_\delta$. As $h \to \Delta$ is a family by assumption, there are local sections $\sigma_i\colon U_i \to h$ with $\sigma_i(\delta) = X_i$, which exponentiate to sections $\tau_i = exp \circ \sigma_i$ through A_i as exp is smooth. Using that multiplication is smooth on the entire family $\mathcal{G}$, the product of these is a smooth section $\tau = \prod_{i=1}^n \tau_i$ defined on the neighborhood $\delta \in \cap_i U_i$. Evaluating at δ shows $\tau(\delta) = A$ and so $\pi\colon \langle exp(h)\rangle \to \Delta$ admits local sections. This shows that *abstractly*, $\mathcal{H}$ is a family over the same base, as it admits local sections. To get that its a subfamily, USE THAT ITS CLOSED.

In the case that $\mathcal{H}$ has disconnected slices, we need to slightly modify the argument above to show that the restricted projection continues to admit local sections. Let $A \in \mathcal{H}_\delta$, and let B be a point in the same component lying in the image of a section $\sigma\colon U \to \mathcal{H}$. Then $B^{-1}A$ is in the connected component of the identity, and so by the previous proposition there is a section $\tau\colon V \to \mathcal{H}$ through $B^{-1}A$. Multiplying by the section through B gives a section $\sigma \cdot \tau\colon U \cap V \to \mathcal{H}$ through A. □

The additional hypothesis that each component of each fiber group contains at least one point contained in the image of a local section may seem rather contrived, but it is quite common and easily checkable in practice. In particular, when considering conjugacy limits there are *global* sections through any points of the original group invariant under the conjugation action.

5 Families of Homogeneous Geometries

To define homogeneous geometries, its clear what we want: some sort of homogeneous space object in Fam_B. To see what these are, we need to define the action of a family of a fibered manifold.

5.1 Group Actions, Stabilizers and Quotients

Families of groups are Lie groupoids, so this is just a certain action of a Lie groupoid on a fibered manifold. In trying to build everything as an exact analog to the smooth category, its important that this choice aligns exactly.

Definition 27: *An action of $\mathcal{G}$ on $\mathcal{X}$ in Fam_B is a morphism $\alpha\colon \mathcal{G}\times_B\mathcal{X} \to \mathcal{X}$ satisfying the axioms for a group action on all local sections.*

In practice we surpress α and write $g.x = \alpha(g,x)$. Spelling out the defining condition above in terms of local sections, for any two sections $g, h\colon U \to \mathcal{G}$ and any $x\colon U \to \mathcal{X}$, we have $g(-).(h(-).x(-)) = (g(-)h(-)).x(-)$ as maps $U \to \mathcal{X}$ and $e(-).x(-) = x(-)$.

A group action is *proper* if the map $\mathcal{G}\times_B\mathcal{X} \to \mathcal{X}\times_B\mathcal{X}$ sending (g,x) to $(g.x, x)$ is proper. An action is *free* if $g.x = x$ implies $g \in e(B)$; or equivalently the actions $G_b \curvearrowright X_b$ are free for all $b \in B$. As an example that will be important later, we consider the action of a family of subgroups $\mathcal{H} < \mathcal{G}$ on $\mathcal{G}$ by translation.

Proposition 15: *Let $\mathcal{G} \to B$ be a family of groups and $\mathcal{H} < \mathcal{G}$ a subgroup family. Then the action of $\mathcal{H}$ on $\mathcal{G}$ by $h.g = hg$ is free and proper.*

Proof. The action is free as each $h \in H_b$ acts by translation on G_b. We need to show that the corresponding map $\alpha : \mathcal{G}\times_B\mathcal{H} \to \mathcal{G}\times_B\mathcal{G}$ given by $(g,h) \mapsto (g, gh)$ is a proper map. Let $K \subset \mathcal{G}\times_B\mathcal{G}$ be compact with $\alpha^{-1}(K) = \{(g,h) \in \mathcal{G}\times_B\mathcal{H} \mid (g,gh) \in K\}$. Choose a sequence $(g_i, h_i) \in \alpha^{-1}(K)$, then $(g_i, g_ih_i) \in K$ subconverges $(g_{i_k}, g_{i_k}h_{i_k}) \to p$. Projecting onto each factor shows $g_{i_k} \to g_\infty$ and $g_{i_k}h_{i_k} \to k$ and so $p = (g_\infty, k) \in K$.

Inversion is a morphism $\mathcal{G} \to \mathcal{G}$, so $g_{i_k}^{-1}$ converges to g_∞^{-1}, and $(g_{i_k}^{-1}, g_{i_k}h_{i_k})$ converges in $\mathcal{G}\times_B\mathcal{G}$ to (g_∞^{-1}, k). But multiplication is a morphism so $\mu(g_{i_k}^{-1}, g_{i_k}h_{i_k}) = g_{i_k}^{-1}g_{i_k}h_{i_k} = h_{i_k}$ converges to $h_\infty = g_\infty^{-1}k \in \mathcal{G}$. As $\mathcal{H}$ is a subfamily, it is closed and $h_\infty \in \mathcal{H}$. Thus, $(g_{i_k}, h_{i_k}) \to (g_\infty, h_\infty) \in \mathcal{G}\times_B\mathcal{H}$. But in fact $\alpha(g_\infty, h_\infty) = (g_\infty, g_\infty h_\infty) = (g_\infty, g_\infty g_\infty^{-1}k) = (g_\infty, k) \in K$ so $(g_\infty, h_\infty) \in \alpha^{-1}(K)$. Thus this space is sequentially compact, and hence compact as the total space / base, being smooth manifolds, are metrizable. ◻

As another instance of the observation that the correct generalziation of 'points' in Fam_B really is local sections of the family projection maps, we generalize the familiar statement that if $G \curvearrowright X$ then an element $g \in G$ induces a homeomorphism $X \to X$.

Proposition 16: *Let $\mathcal{G} \curvearrowright \mathcal{X}$ in Fam_B and let $\sigma\colon U \to \mathcal{G}$ for $U \subset B$ be a local section. Then the induced map $\widehat{\sigma}\colon \mathcal{X}|_U \to \mathcal{X}|_U$ defined by $\widehat{\sigma}(x) = \sigma(\pi_{\mathcal{X}}(x)).x$ is a diffeomorphism.*

Proof. Let $\sigma : U \to \mathcal{G}$ be a local section of $\mathcal{G} \to B$ and $\mathcal{X}|_U$ the corresponding restriction of $\mathcal{X}$. Then $\widehat{\sigma} \in \mathsf{End}(\mathcal{X}|_U)$ as it is expressible as a composition of morphisms, $\widehat{\sigma}(x) = \alpha(\sigma \circ \pi_{\mathcal{G}}(\cdot)), \cdot)$, so it suffices to show $\widehat{\sigma}$ is invertible. As inversion is given by a morphism $\iota \in \mathsf{End}(\mathcal{G})$, the composition $\iota \circ \sigma$ is a section inducing $\widehat{\iota \circ \sigma} \in \mathsf{End}(\mathcal{X}|_W)$, and $(\widehat{\iota \circ \sigma}\widehat{\sigma})(x) = \iota(\sigma(\delta(x))).\sigma(\delta(x)).x = x$. ◻

The ordinary action of G on a space X restricts to an action on each orbit in X. When these orbits fit together to form a family, this gives an action of G on the family.

Proposition 17: *Let $G \curvearrowright X$ be a group action in the smooth category such that X/G is a smooth manifold, and the projection onto orbits $X \to X/G$ sending $x \mapsto G.x$ is a submersion. Then the G action on X induces an action of the trivial family $G \times (X/G) \to (X/G)$ on the family $X \to X/G$ in $\mathsf{Fam}_{X/G}$.*

Proof. Let $\alpha\colon G \times X \to X$ be the action map and $\mathcal{G} = G \times (X/G)$. Then the map $\widetilde{\alpha}\colon \mathcal{G} \times_{X/G} X \to X$ defined by $\widetilde{\alpha}((g, O), x) = \alpha(g, x)$ is a morphism of families as $\pi((g, O), x) = O$ implies $x \in O$ and so $gx \in O$ lies in the same G orbit, thus $\pi_O\widetilde{\alpha}((g, O), x) = O$ so $\pi_O\widetilde{\alpha} = \pi$. But $\widetilde{\alpha}$ obviously satisfies the axioms of a group action fiberwise, as it is just the original action of G restricted to a single orbit. ◻

We will use this to construct our first example of a new transition: from hyperbolic to de-Sitter geometry in CITE SECTION.

If $\mathcal{G} \curvearrowright \mathcal{X}$ is an action of families over B, for each $b \in B$ and each $x_b \in X_b$, the stabilizer subgroup $\mathsf{stab}_{G_b}(x_b) < G_b$ consists of all elements fixing x_b. Stabilizers play an important role in the theory of families of homogeneous spaces to come, so we deal with some of the subtleties now.

Definition 28: *Let $\mathcal{G} \curvearrowright \mathcal{X}$. The* stabilizer *of this action is the union of all point stabilizers, $\mathsf{stab}_{\mathcal{G}} = \bigcup_{x \in \mathcal{X}} \mathsf{stab}_{G_{\pi(x)}}(x)$, topologized as a subset of $\mathcal{G} \times \mathcal{X}$, the trivial $\mathcal{G}$ family over $\mathcal{X}$.*

Note that we have been careful to say *subset* instead of *subfamily*: it is not true that the stabilizers of an action of families be a subfamily. Indeed - consider the standard projective action of $\mathsf{SO}(2, 1)$ on $\mathbb{RP}^2$: point stabilizers of points in $\mathbb{H}^2$ are isomorphic to $\mathsf{SO}(2)$, and points in the interior of its complement have $\mathsf{SO}(1, 1)$ stabilizer - both are 1-dimensional. But, points on the boundary $\partial_\infty\mathbb{H}^2$ have 2-dimensional stabilizer, and thus $\mathsf{stab}_{\mathsf{SO}(2,1)}$ cannot be a subfamily of $\mathsf{SO}(2, 1) \times \mathbb{RP}^2 \to \mathbb{RP}^2$ as its 'members' have differing dimensions. In this example, the jump in dimension in stabilizer is actually caused by a collapse in dimension of the lightcone under projectivization: if instead we computed the point stabilizers of the $\mathsf{SO}(2,1)$ action on $\mathbb{R}^3 \setminus \{0\}$, we see in Chapter CITE that these form a smooth family.

However, under many conditions of practical importance the stabilizers of an action do form a family. Most notably for our purposes, it suffices to require that the action of $\mathcal{G}$ on $\mathcal{X}$ is transitive on each member.

Proposition 18: *Let $\mathcal{G} \curvearrowright \mathcal{X}$ be such that for each $b \in B$, the action $G_b \curvearrowright X_b$ is transitive. Then $\mathsf{stab}_{\mathcal{G}}$ is a smooth subfamily of $\mathcal{G} \times \mathcal{X} \to \mathcal{X}$.*

Proof. We may construct the stabilizer of the action $\mathsf{stab}_{\mathcal{G}} = \{(g,x) \mid g.x = x\}$ as a pullback: namely the pullback of $\mathcal{G} \times_B \mathcal{X} \to \mathcal{X} \times_B X$ along the diagonal $\delta\colon \mathcal{X} \to \mathcal{X} \times_\Delta \mathcal{X}$ has total space $\{((g,x),y) \mid (x,gx) = (y,y)\} \cong \mathsf{stab}_{\mathcal{G}}$.

$$\begin{array}{ccc} \delta^\star(\mathcal{G} \times_B \mathcal{X}) & \dashrightarrow & \mathcal{G} \times_B \mathcal{X} \\ \downarrow & & \downarrow \alpha \\ \mathcal{X} & \xrightarrow{\ \delta\ } & \mathcal{X} \times_B \mathcal{X} \end{array}$$

Thus, $\mathsf{stab}_{\mathcal{G}}$ is a family over $\mathcal{X}$ if the pullback $\delta^\star(\mathcal{G} \times_B \mathcal{X})$ exists, and by CITE this follows immediately if $\alpha\colon \mathcal{G} \times_B \mathcal{X} \to \mathcal{X} \times_B \mathcal{X}$ is a family, i.e. that α is a submersion. Since α is a smooth map of families by CITE, it is a submersion if $\alpha|_b\colon G_b \times X_b \to X_b \times X_b$ is a submersion for each member. But this in turn follows a general fact: given any Lie group G acting transitively on a smooth manifold X, the action $\alpha\colon G \times X \to X \times X$ is a submersion, which we now prove.

Fix a particular $(g,x) \in G \times X$. As the tangent space to the image decomposes as a product $T_{(x,gx)}X \times X = T_xX \times T_{gx}X$, it is enough to show that $d\alpha_{(g,x)}$ is onto each factor. Fixing g, we consider the restricted map $\alpha(g,\cdot)\colon \{g\} \times X \to X \times X$ sends x to (gx,x), and so the derivative is the graph of L_g (left multiplication by g) in $T_xX \times T_{gx}X$. Fixing x, we consider the map $\alpha(\cdot,x)\colon G \times \{x\} \to X \times X$, which is constant on the first factor and is the orbit map $G \to X$, $g \mapsto g.x$ on the second. This map factors through the projection onto the coset space $G \to G/\mathsf{stab}(x)$ to a diffeomorphism $G/\mathsf{stab}(x) \to X$ as the action is transitive. But the projection onto the coset space is a submersion by the quotient manifold theorem, so $\alpha(\cdot,x)$ is onto $\{0\} \times T_{gx}X$. Noting that $\{(v, L_g(v)) \mid v \in T_xX\}$ and $\{(0,w) \mid w \in T_xX\}$ sum to all of $T_{(x,gx)}X \times X$ finishes the argument.

□

An action of families $\mathcal{G} \curvearrowright \mathcal{X}$ gives rise to the *orbit relation* on $\mathcal{X}$ where $x \sim x'$ if $g.x = x'$ for some $g \in \mathcal{G}$. The quotient is the *orbit space* $\mathcal{X}/\mathcal{G}$. This orbit space can be badly behaved in general, and so it is of interest to determine which actions have reasonable quotients. A result of great importance to us is the Quotient Family Theorem, which gives sufficient conditions for the quotient $\mathcal{X}/\mathcal{G}$ to be a family in Fam_B.

Theorem 7 (Quotient Family Theorem)**:** *Let $\mathcal{G} \curvearrowright \mathcal{X}$ be a proper free action in* Fam_Δ*. Then $\mathcal{X} \xrightarrow{\pi} \Delta$ factors as $\mathcal{X} \xrightarrow{\pi_\mathcal{O}} \mathcal{X}/\mathcal{G} \xrightarrow{\overline{\pi}} \Delta$ with $\mathcal{X}/\mathcal{G} \to \Delta$ in* Fam_Δ*, as a family of families $\pi_\mathcal{O} \colon \mathcal{X} \to \mathcal{X}/\mathcal{G}$ and $\overline{\pi} \colon \mathcal{X}/\mathcal{G} \to \Delta$.*

Restricting to the case $B = \{*\}$, this recovers the Quotient Manifold theorem: that the quotient of a manifold by a smooth proper free action of a Lie group is again a smooth manifold. The author would like to thank Jesse Wolfson for pointing out that this may be deduced from general results about Lie groupoid actions.

Proof. FILL IN ARGUMENT ABOUT CLOSED EQUIVALENCE RELATION BY A LIE GROUPOID □

However, a more direct proof adapting the style of argument for the classical quotient manifold theorem from $\mathsf{Diff} = \mathsf{Fam}_{\{*\}}$ to Fam_B may also be given, which can be found in the author's thesis CITE.

Sketch. We utilize a specific atlas of charts on $\mathcal{X}$ which are nicely suited to study the action of $\mathcal{G}$. We call a chart (U, ϕ) on $\mathcal{X}$ to be $\mathcal{G}$-adapted $\phi \colon U \to I^k \times I^\ell \times I^m$ is a homeomorphism onto a cube and (1) the fibers of $\pi_\mathcal{X}$ are precisely $\{x\} \times I^{\ell+m}$ in coordinates for fixed $x \in I^k$ and (2) the fibers of the orbit projection $\mathcal{X} \to \mathcal{X}/\mathcal{G}$ are precisely $\{x, y\} \times I^m$ for fixed $x, y \in I^{k+\ell}$.

The existence of $\mathcal{G}$-adapted charts follows from analyzing the action of $\mathcal{G}$. Let $\mathcal{X}/\mathcal{G} = \{G_b.x \mid x \in X_b, b \in B\}$ be the collection of all orbits. Because $\mathcal{G} \curvearrowright \mathcal{X}$ is an action of families, the members G_b are equidimensional, and as the action is free, the orbits of G_b are all diffeomorphic to G_b, and foliate X_b. Letting $\mathcal{T}$ be the distribution of tangent planes to the orbits $\mathcal{O}$, we use the Frobenius theorem of smooth topology to note that since $\mathcal{T}$ is integrable, there are *flat charts* for $\mathcal{T}$ on $\mathcal{X}$, that is, charts where the submanifolds of $\mathcal{X}/\mathcal{G}$ are locally $\{z\} \times I^m$ in coordiantes. A similar argument shows that there are flat charts with respect to the collection of members $\{X_b \mid b \in B\}$, and a little more work shows we may use these to product charts which are flat with respect to both collectuions simultaneously: that is, $\mathcal{G}$-adapted.

These charts allow us to immediately conclude that the orbit space $\mathcal{X}/\mathcal{G}$ is locally Euclidean, and thus a topological manifold of dimension $\dim \mathcal{X} - \dim \mathcal{G}/\dim B$. Noting that $\mathcal{G}$-adapted charts (U, ϕ) on $\mathcal{X}$ naturally induce charts on $\mathcal{X}/\mathcal{G}$ by quotienting the domain by the orbit relation and projecting of the final I^m factor of the codomain, we show these induced charts give $\mathcal{X}/\mathcal{G}$ the structure of a smooth manifold. The rest is immediate given these coordinates: the projection on to orbits is locally the projection $I^{k+\ell+m} \to I^{k+\ell}$, and the projection from orbits onto the base manifold B is locally $I^{k+\ell} \to I^k$. Both of these are subersions, and so both $\mathcal{X} \to \mathcal{X}/\mathcal{G}$ and $\mathcal{X}/\mathcal{G} \to B$ are smooth families of manifolds.

□

5.2 Homogeneous Space Objects in Fam_B

Using the definitions and work above, we may now define the homogeneous spaces internal to Fam_B.

Definition 29: *A family of geometries parameterized by a smooth manifold B is a homogeneous space object in Fam_B. That is, a triple $(\mathcal{G}, \mathcal{X}, \alpha)$ of families of groups, manifolds $\mathcal{G}, \mathcal{X} \in \mathsf{Fam}_B$ equipped with a surjective action map $\alpha\colon \mathcal{G} \times_B \mathcal{X} \to \mathcal{X} \times_B \mathcal{X}$. A morphism of families of geometries is a pair of smooth maps $(\Phi, F)\colon (\mathcal{G}, \mathcal{X}, \alpha) \to (\mathcal{H}, \mathcal{Y}, \beta)$ respecting the actions: that is, Φ is a family homomorphism and for all local sections $g\colon U \to \mathcal{G}$, $x\colon U \to \mathcal{X}$ we have $F \circ \alpha(g(-), x(-)) = \beta(\Phi \circ g(-), F \circ x(-))$ as maps $U \to \mathcal{Y}$.*

The surjectivity of $\alpha\colon \mathcal{G}\times_B\mathcal{X} \to \mathcal{X}\times_B\mathcal{X}$ is equivalent to the action being *transitive on members*: that for each $b \in B$ the pair (G_b, X_b) equipped with group action $\alpha|_b$ is a homogeneous space. We will continue to supress the action from notation when it does not lead to confusion. Subfamilies, and fiberings of families of geometries are also defined analogously to the original category.

Definition 30: *A family of subgeometries of $(\mathcal{G}, \mathcal{X})$ is an embedding $\iota\colon (H, Y) \hookrightarrow (G, X)$. Conflating such a map with its image, a subgeometry is a family of closed subgroups $H < G$ acting memberwise transitively on a subfamily $\mathcal{Y} \hookrightarrow X$. In this situation we also call $(\mathcal{G}, \mathcal{X})$ a* containing family *for $(\mathcal{H}, \mathcal{Y})$.*

Definition 31: *A fibration of families $(\mathcal{G}, \mathcal{X})$ over $(\mathcal{H}, \mathcal{Y})$ is a pair consisting of a submersion of spaces $\mathcal{X} \to \mathcal{Y}$ equivariant with respect to a submersion of families of groups $\mathcal{G} \to \mathcal{H}$.*

Definition 32: *Let $(\mathcal{G}, \mathcal{X})$ and $(\mathcal{H}, \mathcal{X})$ be two families of geometries with the same underlying family of spaces $\mathcal{X}$. Then $(\mathcal{H}, \mathcal{X})$ is a strictification of $(\mathcal{G}, \mathcal{X})$ if there is a monomorphism $(\Phi, \mathrm{id}_{\mathcal{X}})\colon (\mathcal{H}, \mathcal{X}) \hookrightarrow (\mathcal{G}, \mathcal{X})$. Conversely, $(\mathcal{G}, \mathcal{X})$ is a relaxation of $(\mathcal{H}, \mathcal{Y})$ in this circumstance.*

5.3 Equivalence of Categories

The reason for all this careful building of all definitions to be direct analogs of the smooth category offers some payout here, where we can show the theory of *families of homogeneous spaces* mimics very closely the classical theory of homogeneous spaces itself. To build the analogs of the usual desriptions in Section 3, we develop pointed homogeneous spaces.

Definition 33: *A family of pointed geometries $(\mathcal{G}, \mathcal{X}, x)$ is a family of geometries $(\mathcal{G}, \mathcal{X})$ together with a choice of global section $x\colon B \to \mathcal{X}$. Morphisms $(F, \Phi)\colon (\mathcal{G}, (\mathcal{X}, x)) \to (\mathcal{H}, \mathcal{Y}, y)$ must respect this choice of basepoint by $F \circ x = y$.*

Again, the global section $x\colon B \to \mathcal{X}$ picking a basepoint for each member geometry canonnically picks out a point stabilizer as well.

Definition 34: *A family of geometries (encoded by their automorphism / stabilizers) is a pair $(\mathcal{G}, \mathcal{K})$ of a family of lie groups $\mathcal{G} \in \mathsf{Fam}_B$ together with a closed subfamily $\mathcal{K} < \mathcal{G}$.*

Like the usual theory of homogeneous spaces, there is an equivalence here, allowing us to move freely from one description to the other.

Lemma 19: *The assignements* $\mathcal{F}_1\colon (\mathcal{G},(\mathcal{X},x)) \to (\mathcal{G},\mathsf{stab}_{\mathcal{G}}(x))$ *and* $\mathcal{F}_2\colon (\mathcal{G},\mathcal{K}) \mapsto (\mathcal{G},(\mathcal{G}/\mathcal{K},\mathcal{K}))$ *are functorial.*

Proof. Let $(\mathcal{G},\mathcal{X},x)$ be a pointed geometry. Then by CITE the collection $\mathsf{stab}_{\mathcal{G}}$ of all point stabilizers is a family of groups over $\mathcal{X}$, and we may use the section of basepoints $x\colon B \to \mathcal{X}$ to pull this back to a family $x^\star\mathsf{stab}_{\mathcal{G}}$ containing the stabilizer of each basepoint. CITE implies the projection onto the first factor is an embedding, so we may take $x^\star\mathsf{stab}_{\mathcal{G}}$ to be a closed subfamily of $\mathcal{G} \to B$. Thus $(\mathcal{G}, \mathcal{stab}_{\mathcal{G}}(x))$ is a geometry of the group-stabilizer variety. Recalling that a morphism $\Phi : (\mathcal{G},(\mathcal{X},x)) \to (\mathcal{H},(\mathcal{Y},y))$ consists of a group homomorphism Φ_{Grp} and an equivariant map Φ_{Sp} between the spaces, the image $\Psi(\Phi) = \Phi_{\mathsf{Grp}}$ is simply the group homomorphism, which is well-defined as $\Phi_{\mathsf{Sp}} \circ x = y$ together with equivariance implies that $\Phi_{\mathsf{Grp}}(\mathcal{stab}(x)) \subset \mathcal{stab}(y)$.

We now turn to $\mathcal{F}_2$ applied to a geometry $(\mathcal{G},\mathcal{K})$. As $\mathcal{K}$ is a closed subfamily of $\mathcal{G}$, ?? shows that left translation by $\mathcal{K}$ on $\mathcal{G}$ is a free and proper action. Thus by the quotient family theorem, $\mathcal{G}/\mathcal{K}$ is a smooth family over B. The action of $\mathcal{G}$ on $\mathcal{G}/\mathcal{K}$ is just the usual action of $\mathcal{G}$ on itself followed by the quotient map, which is fiberwise transitive as G_b acts transitively on itself. The natural inclusion $\mathcal{K} \hookrightarrow \mathcal{G}/\mathcal{K}$ (equivalently, the projection of the identity section e) provides the section of points. Given a morphism $\Phi : (\mathcal{K},\mathcal{H}) \to (\mathcal{C},\mathcal{G})$ we define $\mathsf{F}(\Phi) = (\Phi,\overline{\Phi})$ where $\overline{\Phi}(g\mathcal{C}_\delta) = \Phi(g)\mathcal{K}_\delta$. This is Φ-equivariant and well-defined as $\Phi(\mathcal{C}) \subset \mathcal{K}$. Note $\overline{\Phi}(\mathcal{C}) = \mathcal{K}$ as required.

□

Completing the story, these allow us to go back and forth between pointed geometries and pairs of families of Lie groups together with closed subgroups at will.

Theorem 8: *There is an equivalence of categories between pointed geometries and families of Lie groups together with families of closed subgroups.*

Proof. The composition $\mathcal{F}_2\mathcal{F}_1$ is the identity on the category of geometries encoded as families of groups together with closed subgroups, and the composition $\mathcal{F}_1\mathcal{F}_2$ takes the geometry $(\mathcal{G},(\mathcal{X},x))$ to $(\mathcal{G},(\mathcal{G}/x^\star\mathsf{stab}_{\mathcal{G}},x^\star\mathsf{stab}_{\mathcal{G}}))$. Thus it suffices to construct a natural transformation from the identity on the category of pointed families to $\mathcal{F}_1\mathcal{F}_2$. For each fixed geometry $(\mathcal{G},(\mathcal{X},x)$ we define $\eta_{(\mathcal{G},(\mathcal{X},x))}\colon (\mathcal{G},(\mathcal{X},x)) \to \big(\mathcal{G},(\mathcal{G}/x^\star\mathsf{stab}_{\mathcal{G}},x^\star\mathsf{stab}_{\mathcal{G}})\big)$ as $\eta_{(\mathcal{G},(\mathcal{X},x))} = (\mathsf{id}_{\mathcal{G}},\xi_{(\mathcal{G},(\mathcal{X},x))})$ to be the identity on the family of groups, and defined on the family of spaces by the rule $\xi_{(\mathcal{G},(\mathcal{X},x))}(p,x(b)) = g.\mathsf{stab}_{G_b}(x(b)),\mathsf{stab}_{G_b}(x(b))$ for $p \in X_b$ and $g \in G_b$ chosen such that $\mathsf{stab}_{G_b}(p) = g\mathsf{stab}_{G_b}(x(b))g^{-1}$. Then the collection $\{\eta_{(\mathcal{G},(\mathcal{X},x))}\}$ ranging over all pointed geometries is a natural transformation from $\mathcal{F}_1\mathcal{F}_2$ to the identity.

To see this it suffices to check that $\overline{\Phi_{\mathsf{Grp}}} \circ \xi_{(\mathcal{G},\mathcal{X})} = \xi_{(\mathcal{H},\mathcal{Y})} \circ \Phi_{\mathsf{Sp}}$. Let $p \in \mathcal{X}_\delta$ and $g \in \mathcal{G}_\delta$ be such that $g.x(\delta) = p$. Then $\xi_{(\mathcal{G},\mathcal{X})}(p) = g\mathsf{stab}_{\mathcal{G}}(x(b))$ and $\overline{\Phi_{\mathsf{Grp}}}(g\mathsf{stab}_{\mathcal{G}}(x(b))) = \Phi_{\mathsf{Grp}}(g)\mathsf{stab}_{\mathcal{H}}(y(b)))$. Computing the other way around we find $\Phi_{\mathsf{Sp}}(p) = \Phi_{\mathsf{Sp}}(g.x_b) = \Phi_{\mathsf{Grp}}(g)\Phi_{\mathsf{Sp}}(x_b) = \Phi_{\mathsf{Grp}}(g)y_b$ and $\xi_{(\mathcal{H},\mathcal{Y})}(\Phi_{\mathsf{Grp}}(g)y_b)) = \Phi_{\mathsf{Grp}}(g)\mathsf{stab}_{\mathcal{H}}(y_b)$.

$$\begin{array}{ccc}(\mathcal{G},(\mathcal{X},x)) & \xrightarrow{(\mathsf{id}_{\mathcal{G}},\xi_{(\mathcal{G},\mathcal{X})})} & \left(\mathcal{G},(\mathcal{G}/x^\star\mathsf{stab}_{\mathcal{G}},x^\star\mathsf{stab}_{\mathcal{G}})\right)\\ {\scriptstyle(\Phi_{\mathsf{Grp}},\Phi_{\mathsf{Sp}})}\downarrow & & \downarrow{\scriptstyle(\Phi_{\mathsf{Grp}},\overline{\Phi_{\mathsf{Grp}}})}\\ (\mathcal{H},(\mathcal{Y},y)) & \xrightarrow[(\mathsf{id}_{\mathcal{H}},\xi_{(\mathcal{H},\mathcal{Y})})]{} & \left(\mathcal{H},(\mathcal{H}/y^\star\mathsf{stab}_{\mathcal{H}},y^\star\mathsf{stab}_{\mathcal{H}})\right)\end{array}$$

□

Even analogs of the other constructions from Section 3: Kernels of geometry action exist and these are smooth subfamilies! Thus we can quotient.

Proposition 20: *Effectivization is a natural transformation*

Proof. □

Thus, when considering transitional problems we are free to be loose with our choice of effective vs noneffective models: we may build whatever transition is easiest and then effectivize the entire construction after the fact if desired. However, much like when dealing with homogeneous geometries themselves, we won't worry about this sort of thing. In the end we only care about geometries up to equivalence: their effectivizations are locally isomorphic. So, for example, we will say we have constructed a transition of spherical geometry whenever we find a transitioning family contianing some member equivalent to $\mathbb{S}^n$.

5.4 Example: The Transition of Constant Curvature Geometries

As a first example of these definitions, we formalize the familiar transition from hyperbolic to spherical geometry through Euclidean. First, we construct the family of spaces.

Lemma 21: *Let $\mathcal{V} = V\left(x_0\sum_{i=1}^{n-1}x_i^2 + x_n^2 - 1\right) \subset \mathbb{R}^{n+1}$. Then the restriction of projection onto the x_0-axis equips V with the structure of a family of spaces over $\mathbb{R}$.*

Proof. $\mathcal{V}$ is a smooth submanifold of $\mathbb{R}^{n+1}$ with normal vector given by $\nabla\left(x_0\sum_{i=1}^{n-1}x_i^2 + x_n^2 - 1\right) = (1, 2x_0x_1, \ldots, 2x_0x_n)$ is nowhere parallel to the x_0 axis, so the tangent spaces to $\mathcal{V}$ are transverse to the foliation $\{x_0\}\times\mathbb{R}^n$, and the restricted projection is a submersion. □

The family $\mathcal{V}$ has members transitioning from hyperboloids of 2 sheets for $t < 0$ to ellipsoids for $t > 0$ through a pair of parallel planes at $t = 0$. Each of these slices admits a free $\mathbb{Z}_2$ action sending a point to its antipode, and so $\mathcal{V}$ admits a free and proper action of $\mathcal{Z} = \mathbb{Z}_2 \times \mathbb{R} \to \mathbb{R}$.

Corollary 22: *$\mathcal{X} = \mathcal{V}/\mathcal{Z}$ is a smooth family of subsets of $\mathbb{R}\mathsf{P}^n$ over $\mathbb{R}$ transitioning from the hyperboloid of 1 sheet to $\mathbb{R}\mathsf{P}^n$.*

FIGURE HERE

Now we turn to the family of groups. For each $x_0 \neq 0$, the surface $\mathcal{V}_{x_0}$ is a quadratic hypersurface in $\mathbb{R}^n\times\{x_0\}$, and the group of linear transformations preserving it forms the orthogonal group $(\mathsf{diag}(x_0, \ldots, x_0, 1))$.

Proposition 23: *Let $\mathcal{G} \subset \mathsf{GL}(n+1,\mathbb{R}) \times \mathbb{R}$ be the collection of groups*

$$\mathcal{G} = \bigcup_{t\in\mathbb{R}_-} \mathsf{SO}(\mathsf{diag}(t,\ldots,t,1)) \times \{t\} \cup \mathsf{Euc}(n) \times \{0\} \cup \bigcup_{t\in\mathbb{R}_+} \mathsf{SO}(\mathsf{diag}(t,\ldots,t,1)) \times \{t\}$$

Then $\mathcal{G}$ is a family of groups equipped with the restricted projection from $\mathsf{GL}(n+1;\mathbb{R}) \times \mathbb{R} \to \mathbb{R}$.

Proof. Applying the contragredient automorphism $A \mapsto A^{-T}$ to each member of $\mathsf{GL}(n+1;\mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ gives a smooth automorphism of the family, taking $\mathcal{G}$ to $\mathcal{G}^{-T} = \bigcup_{t\in\mathbb{R}} G_t^{-T} \times \{t\}$. We show that this collection forms a family directly; and then applying once more then contragredient automorphism gives the same result for the original $\mathcal{G}$. The reason for this seemingly strange approach is just notational: $G_t^{-T} = \mathsf{SO}(t,t,1)^{-T} = \mathsf{SO}(1,1,t)$ for $t \neq 0$, and when $t = 0$ the group $\mathsf{SO}(1,1,0)$ is precisely the contragredient Euclidean group $\mathsf{Euc}(2)^{-T}$, and so the entire family $\mathcal{G}^{-T}$ can be succinctly described as $\mathcal{G}_t^{-T} = \mathsf{SO}(\mathsf{diag}(1,1,t))$ regardless of the value of t (this does not hold in the original case, as $\mathsf{SO}(\mathsf{diag}(0,0,1))$ strictly contains the Euclidean group). Now $\mathcal{G}^{-T}$, is a nonsingular subvariety of $\mathsf{M}(3;\mathbb{R}) \times \mathbb{R}^1$ and thus a closed smooth submanifold.

The Lie algebras $\mathfrak{so}(\mathsf{diag}(1,1,1,t))$ form a continuous family of Lie algebras in $\mathsf{M}(3;\mathbb{R}) \times \mathbb{R}$ as follows immediately from computation. And while the number of components of $\mathsf{SO}(\mathsf{diag}(1,1,t))$ changes along the transition (from 2 when $t < 0$ to 1 when $t > 0$), each component always contains one of the matrices of the form $\mathsf{diag}(\pm1,\pm1,\pm1)$, so by Proposition 14, $\mathcal{SO}(\mathsf{diag}(1,1,t))$ is a family of groups. Thus so is its contragredient image, $\mathcal{G}$. □

The action of G_t on $\mathcal{X}_t$ is transitive, and so $(\mathcal{G},\mathcal{X})$ is a family of Klein geometries. To get a family of pointed geometries, it suffices to choose any $\gamma\colon \mathbb{R} \to \mathbb{R}\mathsf{P}^2$ such that $\gamma(t) \in \mathcal{X}_t$; for instance $\gamma(t) = [0:0:1]$.

5.5 Other Categories of Families

In the applications to follow we focus solely on the smooth category, and the theory presented above. However because of the nature of its development (we built a category Fam_B from Diff, and then developed geometry internal to Fam_B), one could imagine doing this with alternative topological categories to start with, and describing for instance the theory of *continuous families* in Top and so on. For notions stronger than smooth category we restrict the allowable morphisms and build the appropriate subcategories of Fam_B, Grp_B and Geo_B. As an example: to build analytic families of geometries we require analytic base and total spaces, an analytic submersion $\mathcal{X} \to B$ and analytic moprphisms of families, groups.

However, to develop a notion in some larger topological category, we need to generalize the notion of a submersion. This leads naturally to multiple possibilities, as

[1] $\mathcal{G}^{-T} = \bigcup_{t\in\mathbb{R}} \mathsf{SO}(\mathsf{diag}(1,1,t)) \times \{t\}$ is cut out by the equations $(X^T\mathsf{diag}(1,1,t)X, t) = (\mathsf{diag}(1,1,t), t)$, $\det X = 1$ for $X = (x_{ij})_{1\le i,j,\le 3}$.

”submersion” fractures into multiple concepts in categories such as Top. In particular, in the smooth category a map $f: X \to Y$ is a submerison if and only if it is locally a projection, but also if and only if it admits local sections. Outside of the smooth category these concepts become distinct, but strictly ordered: being locally a projection is stonger than admitting local sections.

In what follows use C-Family to mean ”Continuous family” i.e. a. family in Top, the category of Locally compact Hausdorff spaces, or topological manifolds, and call a family strong or weak depending on if its defining projection satisfies the (stronger) condition of being locally a projection, or the (weaker) condition of admitting local sections.

Definition 35 (Weak C-Family)**:** *A weak C-Family is a triple* $(\mathcal{X}, B, \pi)$ *such that* $\pi: \mathcal{X} \to B$ *admits local sections. That is, for each* $x \in \mathcal{X}$ *there is a neighborhood* $U \ni \pi(x)$ *and a section* $\sigma: U \to \mathcal{X}$ *of* π *such that* $\sigma(\pi(x)) = x$.

Definition 36 (Strong C-Family)**:** *A strong C-Family is a triple* $(\mathcal{X}, B, \pi)$ *such that* $\pi: \mathcal{X} \to B$ *is locally a projection. That is, for each* $x \in \mathcal{X}$ *there is a neighborhood* $V \ni x$ *and a homeomorphism* $\phi: Y \times \pi(V) \to V$ *for some space* Y *such that the composition* $\pi\phi: Y \times \pi(V) \to B$ *is projection onto the second factor.*

In category of topological manifolds, members of weak families do not have to be manifolds; can be manifolds ”with degenerations” Example? Project a saddle shape.

Consequences of these: in TOP or LCH, a cone projected onto its axis is a weak but not strong family. Quotient Family Thm doesn’t provide a means of going from one notion of family to another. Does it for strong families?

6 Comparison with Other Formalisms

The goal of this work is to provide a means of talking about geometric transitions without first choosing an ambient space to work in. We now relate the results of this project to the ususal formalisms, mostly conjguacy limits but also ”bundles of coherent elements”.

First, when is one of our abstract families realized inside of some ambient space?

Definition 37: *A family of geometries* $(\mathcal{H}, \mathcal{Y})$ *over* B *can be realized inside an ambient geometry* (G, X) *if there is an embedding of the family* $(\mathcal{H}, \mathcal{Y})$ *into the constant family* $(G \times B, X \times B) \to B$.

We may view the resulting subfamily as a collection of subgroups / submanifolds parameterized by B, using the maps $B \to \mathsf{Closed}(G)$, $B \to \mathsf{Closed}(X)$ defined by $b \mapsto G_b$ and $b \mapsto X_b$.

6.1 Conjugacy Limits

How to talk about continuity inside an ambient space: use space of closed subgroups.

Given a topological space X, the *hyperspace of closed subsets* is denoted $\mathsf{Closed}(X)$. When X is a compact metric space $\mathsf{Closed}(X)$ inherits a topology from the *Hausdorff metric*, and the resulting topology does not depend on the details of the original metric so we may topologize $\mathsf{Closed}(X)$ in this way for any compact *metrizable* X. When X is not compact this construction runs into difficulties, and there are a variety of topolgies on $\mathsf{Closed}(X)$. As we are particularly interested in closed submanifolds of noncompact manifolds, we may use the fact that the 1-point compactification of any manifold is metrizable to define an extension, the *Chabauty topology*.

Definition 38: *Given a manifold X, the Chabauty topology on $\mathsf{Closed}(X)$ is the subspace topology inherited from $\mathsf{Closed}(X \cup \{\infty\})$ with the Hausdorff topology. It is generated by the subbasis $\mathcal{O}_{K,U}$ indexed by compact K and open U in X: $\mathcal{O}_{K,U} = \{Z \in \mathsf{Closed}(M) \mid Z \cap U \neq \varnothing, Z \cap K = \varnothing\}$.*

The Chabauty topology restricts to the Hausdorff topology on any compact subset of X, and so is the topology of Hausdorff convergence on compact sets. For the argument below, it is useful to give an explicit description of the open sets in this topology.

When G is a topological group, then the collection of closed subgroups of G is a closed subset of the space of all closed subsets. In this case will abuse notation and use $\mathsf{Closed}(G)$ to refer to the space of closed subgroups. This defines a topology on the space of subgeometries of a geometry, thinking of this as a group-stabilizer pair.

Definition 39: *The space of subgeometries of (G, K) is given by $\mathsf{SubGeo}_{(G,K)} = \{(H, C) \mid H \in \mathfrak{C}(G),\ C = H \cap K\}$, topologized as a subset of $\mathsf{Closed}(G) \times \mathsf{Closed}(K)$*

A sequence (H_n, Y_n) of subgeometries of (G, X) is *convergent* if it converges in $\mathsf{SubGeo}_{(G,X)}$. The particular limits of interest here are *conjugacy limits*, as studied by Cooper Danciger and Wienhard in *Limits of Geometries*.

Definition 40: *A continuous path of subgeometries of (G, K) is a continuous map $I \to$ mathsf SubGeo$_{(G,K)}$ from some interval $I \subset \mathbb{R}$. A* conjugacy limit *of (H, C) in (G, K) is a geometry $(L, Z) = \lim_{t\to 1} \gamma(t)$ of a continuous map $[0, 1) \to \mathsf{SubGeo}(G, K)$ defined by $\gamma(t) = (A_t H A_t^{-1}, A_t C A_t^{-1})$ for $A_t : [0, 1) \to G$ continuous and $A_0 = I$.*

Note this equivalent to the definition given in *Limits of Geometries* (CITE) where instead it is phrased for groups together with underlying spaces. This leads to more awkward phrasing requiring that eventually the spaces all have a point in common (we have here given the definition for paths instead of the original sequences).

Definition 41 (Conjugacy Limit: Cooper Danciger & Wienhard)**:** *A path of subgeometries $(H_t, Y_t) < (G, X)$ converges to the subgeometry $(L, Z) < (G, X)$ if H_t converges to L in $\mathsf{Closed}(G)$ and there exists $z \in Z \subset X$ such that for all t sufficiently large $z \in Y_t$. We say that a subgeometry (L, Z) is a* conjugacy limit *(or just* limit*) of (H, Y) in (G, X) if there is a sequence $g_t \in G$ such that the conjugate subgeometries $(g_t H g_t^{-1}, g_t Y)$ converge to (L, Z).*

To see the equivalence, note without loss of generality (by starting the path at a later time $t_0 > 0$) we may assume that in fact $z \in Y_t \subset X$ for all t. Then we define$K = \mathsf{stab}_G(z)$, $C = \mathsf{stab}_H(z)$, and $C_t = \mathsf{stab}_{H_t}(z)$. Then (H_t, C_t) is a subgeometry of (G, K) for all t, and as $t \to \infty$ the stabilizing subgroup $C_t = g_t C g_t^{-1}$ converges (as a sequence of subgroups of a convergent sequence of groups) to the limiting stabilizer of z under the action of L. Thus $(H, C) = (H, \mathsf{stab}_H(z))$ converges under g_t conjugacy to $(L, \mathsf{stab}_L(z))$.

The space of closed subgroups of a group is compact, but the space of subgeometries is not: can "crush a subgeometry to a point" for example...

If a family $\mathcal{X} \to B$ embeds in Y, then the members of $\mathcal{X}$ vary continuously in the space of closed subsets of Y.

Proposition 24: *Let $\pi\colon \mathcal{X} \to B$ be a subfamily of $Y \times B \to B$ and $\mathsf{pr}\colon Y \times B \to B$ be projection $(y, b) \to y$ onto the first factor. Then the induced map sending $b \mapsto X_b \subset Y$ for $X_p = \mathsf{pr}\pi^{-1}\{b\}$ is continuous as a map $B \to \mathsf{Closed}(Y)$.*

Proof. First, we show that the map sending b to its fiber $\pi^{-1}\{b\}$ is continuous into $\mathsf{Closed}(\mathcal{X})$. Note that as $\pi\colon \mathcal{X} \to B$ is a submersion it is open. Let $\mathcal{O}_{K,U}$ be a subbasic open set for the Chabauty topology on $\mathcal{X}$ and $\delta \in \pi_*^{-1}\{\mathcal{O}_{K,U}\}$ for $K \subset \mathcal{X}$ compact, $U \subset \mathcal{X}$ open. As $\pi(K)$ is a compact subset of Δ not containing δ there is some open $V \ni \delta$ disjoint from $\pi(K)$. As π is open, $W = V \cap \pi(U)$ is an open neighborhood of δ. Note $W \subset \pi_*^{-1}\left\{\mathcal{O}_{K,U}\right\}$ as if $\eta \in W$ then $\eta \notin \pi(K)$ so $K \cap \pi_*(\eta) = \varnothing$ and $\eta \in \pi(U)$ so $\pi_*(\eta) \cap U \neq \varnothing$. Thus $\pi_*^{-1}\left\{\mathcal{O}_{K,U}\right\}$ is open and π_* is continuous.

Now $\mathcal{X} \subset Y \times B$ is closed so the inclusion $\mathcal{X} \hookrightarrow Y \times B$ induces a continuous map on their spaces of closed subsets, so $b \to \pi^{-1}\{b\}$ is continuous into $\mathsf{Closed}(Y \times B)$. But $\pi^{-1}\{b\} = X_b \times \{b\}$ for $X_b = \mathsf{pr}\pi^{-1}\{b\}$ as above, so we may factor this continuous assignment as $b \mapsto X_b \mapsto X_b \times \{b\}$ and use that the assignment $X_b \to X_b \times \{b\}$ is a homeomorphism onto its image to conclude that $b \to X_b$ is continuous in $\mathsf{Closed}(Y)$ as desired. □

Applying this to the groups of isometries and stabilzers gives the desired result.

Corollary 25: *If $(\mathcal{H}, (\mathcal{Y}, x))$ is a subfamily of the constant family $(G, (X, x)) \times B \to B$, the induced map $b \mapsto (G_b, (X_b, x))$ is a continuous path of subgeometires in $\mathsf{SubGeo}_{(G,X)}$.*

Proof. Let $K = \mathsf{stab}_G(x)$ and $\mathcal{K} = K \times B$ be the family of stabilizers of the constant family $(G, (X, x)) \times B \to B$. Then $\mathcal{C} = \mathcal{K} \cap \mathcal{H}$ are the corresponding point stabilizers for the family of subgeomeries, and by the proposition above both $B \to \mathsf{Closed}(G)$ given by $b \mapsto H_b$ and $B \to \mathsf{Closed}(K)$ given by $b \mapsto C_b$ are continuous, so $B \mapsto (H_b, C_b)$ is continuous into $\mathsf{SubGeo}_{(G,X)}$. □

Going the other direction, starting with a continuous map $I \to \mathsf{SubGeo}_{(G,K)}$ and producing a family of geometries is not easy to state any general results, as there is no notion of 'smooth' in $\mathsf{SubGeo}_{(G,K)}$. However, all limits arising from smooth conjugacies are families, as we show below. We say a path of conjugate groups $H_t = A_t H A_t^{-1} \in \mathsf{Closed}(G)$ is smooth if A_t is smooth as a function of t into G. Equivalently, H_t is smooth when the assignment of corresponding Lie algebras $t \mapsto \mathfrak{h}_t \in \mathsf{Gr}(\dim h, \mathfrak{g})$ is a smooth map. This lets us define a smooth conjugacy limit.

Definition 42: *A transition from H_1 to H_2 in G through a conjugacy limit L is a pair of conjugacy limits $A_t H_1 A_t^{-1} \to L$ and $B_t H_2 B_t^{-1} \to L$ for $A_t, B_t\colon [0, 1)$. This is transition* *is said to be* smooth in G *if the associated maps* $\alpha(t) = A_t \mathfrak{h}_1 A_t^{-1}$ *and* $beta(t) = B_t \mathfrak{h}_2 B_t^{-1}$ *stitch together to a smooth map* $\gamma\colon [0,2] \mapsto \mathsf{Gr}(\dim H_i, \mathfrak{g})$

$$\gamma(t) = \begin{cases} \alpha(t) & t \in [0,1) \\ \mathfrak{lie}(L) & t = 1 \\ \beta(2-t) & t \in (1,2] \end{cases}$$

Note that in general a continuous path of Lie groups does not correspond to a continuous path of Lie algebras. However when the limit is by conjugacy, and the group involved is algebraic this does hold, see for example CITE CDW. Thus for our purposes, smoothness at the level of Lie algebras is enough to actually construct a smooth family of Lie groups, using the results of CITE PREV SECTION.

Proposition 26: *Let* H_1 *be an* n*-dimensional Lie group which transitions to* H_2 *through* L *in* G *by smooth conjugacy. Then this transition is represented by a smooth family of groups* $\mathcal{H} \to I$ *transitioning from the identity component of* H_1 *to the identity component of* H_2 *through the identity component of* L.

Proof. Let $H(t)\colon I = [0,2] \to \mathsf{Closed}(G)$ be the map picking out the groups in the conjugacy limit, so $H(t) = A_t(H_1)_0 A_t^{-1}$ for $t < 1$, $H(1) = L_0$ and $H(t) = B_t(H_2)_0 B_t^{-1}$ for $t > 1$, X_0 the identity component of X. We build $\mathcal{H} \subset G \times I$ as $\mathcal{H} = \{(H(t), t) \mid t \in I\}$, and aim to show that $\mathcal{H}$ is a subfamily of $G \times I$: that is, $\mathcal{H}$ is a smooth manifold, and the projection $G \times I \to I$ restricts to a submersion on $\mathcal{H}$. First, we see the collection $\mathcal{H} \subset G \times I$ is closed in $G \times I$. Indeed, let (M_i, t_i) be any sequence of elements in $\mathcal{H}$, converging to $(M, t) \in G \times I$. If $t < 1$ (similarly for $t > 1$) then $M_i = A_{t_i} N_i A_{t_i}^{-1}$ and as $A_{t_i} \to A_t$ converges, the sequence N_i of elements of H_1 converges in G, but H_1 is closed so this limit is in H, and $M \in H_t$. If $t = 1$ then the sequence M_i is a convergent sequence of elements of $H(t_i)$, and so the limit lies in the Chabauty limit of the sequence of groups $H(t_i)$, which is L by definition.

By assumption of smooth conjugacy the corresponding map on Lie algebras $\mathfrak{h} = \mathfrak{lie} \circ H\colon I \to \mathsf{Gr}(n, \mathfrak{g})$ is smooth. This smooth map corresponds to a vector bundle $h \to I$ with each fiber a Lie algebra, so by CITE a family of Lie algebras over $[0,2] = I$. Let $\exp\colon \mathfrak{g} \to G$ be the Lie group exponential, abusing notation we similarly denote its extension $\mathfrak{g} \times I \to G \times I$, by $\exp(A, t) = (\exp(A), t)$. We first show that $\mathcal{H} \to I$ admits smooth local sections. Let $M \in \mathcal{H}$, with $\pi(M) = t \in I$. If $t < 1$ we may produce the smooth section $A_s M A_s^{-1}$ for $s \in (t - \varepsilon, t + \varepsilon)$, and similarly for $t > 1$. When $t = 1$ we use the more general argument in PROP CITE: if $M = \exp(X)$ for some $X \in \mathfrak{l}$, choose a smooth section σ of $h \to I$ through X, and note $\exp(\sigma(t), t)$ is a smooth section of $G \times I \to I$ through M, fully contained in $\mathcal{H}$. If M is not an exponential, we may write it as a product of exponentials $M = \exp(X_1) \cdots \exp(X_n)$ as it lies in the connected component of the identity, and the product of sections $\sigma_i\colon I \to h$ through X_i give a smooth section $t \mapsto ((\prod \exp \sigma_i(t)), t)$ through $(M, 0)$.

To complete the argument, we need that $\mathcal{H}$ is actually a smooth submanifold of $G \times I$ (both because a smooth total space is a requirement of being a family, and so that the existence of a smooth section through each point of $\mathcal{H}$ implies the projection

$\mathcal{H} \to I$ is a submersion). In particular, we must show that $\mathcal{H}$ is locally Euclidean, and produce a smooth atlas of charts: both of which can be achieved by extending the style of argument above. Let $(A, t) \in \mathcal{H}$, and assume for the sake of simplicity that $A = \exp(X)$ for some $X \in \mathfrak{h}_t$ near the identity (if not, decompose as a product and follow the previous argument). There is some open set $U \subset \mathfrak{g}$ about the origin on which exp is a diffeomorphism, by our assumption that A is sufficiently near the identity we may assume $X \in U$, and $\exp\colon \mathfrak{g} \times I \to G \times I$ is a diffeomorphism on $U \times I$. Let $V \ni X$ be a small open subset of the total space of the vector bundle $h \to I$. By possibly shrinking V (taking the connected component of its intersection with $U \times I$ containing X) we may assume that $\exp|_V : V \to \mathcal{H}$ is a diffeomorphism onto its image. Thus, $\mathcal{H}$ is locally Euclidean at A. Now that we know $\mathcal{H}$ is a closed submanifold of $G \times I$, we give it the smooth structure such that the inclusion $\mathcal{H} \hookrightarrow G \times I$ is a smooth embedding, and note with this smooth structure the sections constructed above are smooth by definition, so the restricted projection $\mathcal{H} \to I$ remains a submerison.

□

We may apply this to the groups, stabilizer subgroups which appear in the various conjugacy limits of orthogonal groups discussed in CITE LIMITS to see

Corollary 27: *All transitions in CITE LIMITS are smooth subfamilies of projective geometry.*

6.2 "Bundles of Groups"

The second formalism that has been used in geometric transitions is similar in spirit to ours, by constructing a fiber bundle type object whose fibers are geometries. This is done in CITE.

Definition 43: *Coherent elements*

Definition 44: *Bundle of geometries*

Whereas the previous definition was weaker in the sense that it only talks about continuity vs smoothness, this is stronger as it requires analyticity. Smooth families can be seen as a direct generalization of this, getting around the issue of the limit groups "being too big" by stipulations on the total space, instead of on the possible paths of group elements.

Proposition 28: *If $(\mathcal{H}, \mathcal{Y})$ is an analytic subfamily of the constant family $(G, X) \times B \to B$, then it is a family IN THE BUNDLE SENSE.*

Proof. □

Conversely, analytic families directly generalize this bundle construction.

Proposition 29: *Every 'analytic bundle of groups' is an analytic family of geometries.*

Proof. □

7 Applications

We've shown that the language of families generalizes the theories of transitions that have been developed before, but now we want to look for uses. There are two main things we aim to show here: that considered abstractly, there are geometric transitions that do not occur as subgeometries of projective space. and Two, that the idea: take a construction we know how to run for smooth manifolds: but try and run it in the category of families succeeds in many cases.

7.1 TRANSITIONS BETWEEN (PSEUDO)-RIEMANNIAN SYMMETRIC SPACES FOR $\mathsf{O}(p,q)$

Intuitively, the pointed geometry $(G,(X,x))$ is the homogeneous space (G,X) viewed from x, and the question *what does* (G,X) *look like from infinity* can be interpreted as *what pointed limit geometries arise as the basepoint is moved into an end of* X?

From the group-stabilizer viewpoint, its clear for general reasons that a limiting geometry exists. Indeed, the pointed geometries with automorphism group G depend only on the stabilizer $K \leqslant G$ and so can be thought of as points in the Chabauty space $\mathfrak{C}_G$. If (G,X) is such a geometry, and $x_t \in X$ is a path of points leaving every compact set, the corresponding stabilizer groups $K_t = \mathsf{stab}_G(x_t)$ subconverge in $\mathfrak{C}_G$ by compactness to a closed subgroup C, and thus a limiting geometry (G,C).

Restricting our attention to the orthogonal and unitary groups we can concretely understand such limiting geometries and realize them as transitions between pairs of well known classical geometries. A motivating example to keep in mind is the hyperbolic plane $\mathbb{H}^2$ thought of as a subgeometry of $\mathbb{R}\mathsf{P}^2$. The quadratic form defining $\mathbb{H}^2$ has signature $(2,1)$ dividing $\mathbb{R}\mathsf{P}^2$ into the hyperbolic plane and an open Mobius band, separated a circle (the projectivization of the null cone). Much as the action of $\mathsf{SO}(2,1)$ on the disk gives hyperbolic space, its action on Mobius band gives the other projective geometry with automorphism group $\mathsf{SO}(2,1)$, a Lorentzian geometry called *de Sitter space*. Any path of points x_t remaining in the disk give models of hyperbolic space $(\mathsf{SO}(2,1),\mathbb{D}^2,x_t)$ and any points in the Mobius band give models of de Sitter space $(\mathsf{SO}(2,1),\mathsf{Mob},x_t)$. Throughout the rest of this section we focus on families of points crossing between the two.

More generally, if G is any orthogonal or unitary subgroup of $\mathsf{GL}(n;\mathbb{R})$ or $\mathsf{GL}(n;\mathbb{C})$ the associated quadratic / hermitian form defines a positive and negative cone, whose projectivizations X_+ and X_- are the domains for the two projective geometries (G,X_+), (G,X_-) with automorphism group G. The isomorphism type of the geometries depend on the signature (p,q) of the form: X_+ is not isomorphic to X_- unless $p=q$. The main theorem of this section provides a transition between these geometries.

Theorem 9: *There is a transition from* (G,X_+) *to* (G,X_-) *for any orthogonal or unitary group* G.

Proof. Fix an orthogonal or unitary group $G \leqslant \mathsf{GL}(n+1;\mathbb{F})$ and consider its linear action on $\mathbb{F}^{n+1}$. The group preserves a quadratic / Hermitian form J, and the level sets of J are precisely the orbits of G on $\mathbb{F}^{n+1} \smallsetminus \vec{0}$ (the origin is fixed by the linear action). In fact, the map $q_J : \mathbb{F}^{n+1} \smallsetminus \{0\} \to \mathbb{R}$ is a submersion, and gives $\mathbb{F}^{n+1} \smallsetminus \{0\}$ the structure of a family over $\mathbb{R}$ (if J has signature $(n, 0)$ this only maps onto $\mathbb{R}^+$). As these level sets are the G orbits $\mathcal{O} = (\mathbb{F}^{n+1} \smallsetminus \{0\})/G$, we are exactly in the situation of ??, and the action of G on $\mathbb{F}^{n+1} \smallsetminus \{0\}$ induces an action of families $G \times \mathcal{O}$ on $\mathbb{F}^{n+1} \smallsetminus \{0\} \to \mathcal{O}$. This provides a transition from (G, X_-) to (G, X_+) as non-pointed geometries, because the negative level sets of q_J projectivize to X_- and similarly $\mathbb{P}q_J^{-1}(\mathbb{R}_+) = X_+$. ◻

The level sets of q_J foliate the complement of $\vec{0}$, each determining a geometry when equipped with the action of G. The transition occurs passing through the zero level set, which is the null cone of the form (of course, there is no nontrivial transition for signature $(n, 0)$). Thus the geometry (G, X_+) transitions to (G, X_-) through the geometry associated to the G action on the *non-projectivized lightcone* $X_0 = \left\{ v \neq 0 \mid q_J(v) = 0 \right\}$.

Corollary 30: *To each classical orthogonal / unitary group there corresponds a family of pointed geometries with base* $\mathbb{F}\mathsf{P}^n$.

Proof. Given the smooth family of geometries $(G \times \mathbb{R}, \mathbb{F}^{n+1} \smallsetminus \{0\})$ above, ?? shows that the collection of point stabilizers form a family with base the total space of the geometry, here $\mathbb{F}^{n+1} \smallsetminus \{0\}$. Thus we have a family of pointed geometries (given in the point stabilizer formalism) with base $\mathbb{F}^{n+1} \smallsetminus \{0\}$. As the action on $\mathbb{F}^{n+1}$ is linear however, the point stabilizer assigned to x and αx are equal for all $\alpha \in \mathbb{F}^{\times}$, to this descends to a family $\mathcal{stab} \to \mathbb{F}\mathsf{P}^n$. This induces the claimed family of geometries $(G \times \mathbb{F}\mathsf{P}^n, \mathcal{stab})$ in $\mathsf{Fam}_{\mathbb{F}\mathsf{P}^n}$. ◻

The resulting family is *almost* a family of projective geometries, in the sense that for $[x] \in \mathbb{F}\mathsf{P}^n$ with $q_J(x) \neq 0$ the member above $[x]$ is isomorphic to (G, X_+) or (G, X_-). However for points $[x]$ lying on the null cone, the geometry is *not* a projective geometry as the domain is the unprojectivized cone. Thus these are examples of transitions between two projective geometries, which do not occur *through* projective geometries.

The Hyperbolic - de Sitter Transition

In all dimensions, the null cone for the $(n, 1)$ form divides $\mathbb{R}\mathsf{P}^n$ into an n ball and its complement; the action of $\mathsf{SO}(n, 1)$ on $\mathbb{D}^n$ defines the Klein model of $\mathbb{H}^n$ and on the complement a projective model of de Sitter space $\mathsf{d}\mathbb{S}^n$. Here we briefly discuss the transitional geometry in this case. The lightcone of the $(n, 1)$ form projectivizes to $\mathbb{S}^{n-1} \subset \mathbb{R}\mathsf{P}^n$ forming the common boundary to $\mathbb{H}^n$ and $\mathsf{d}\mathbb{S}^n$. The action of $\mathsf{SO}(n, 1)$ on $\mathbb{S}^{n-1}$ determines a model of conformal geometry (the isometries of hyperbolic space determine conformal transformations of the ideal boundary), and so realizing the null cone as the canonical line bundle to the projective $\mathbb{S}^{n-1} \subset \mathbb{R}\mathsf{P}^n$, the light cone geometry is just the geometry of the canonical line bundle to the conformal sphere.

Corollary 31: *There is a transition from* $\mathbb{H}^n$ *to* $\mathsf{d}\mathbb{S}^n$ *through the geometry of the canonical line bundle to the conformal* $n - 1$ *sphere.*

7.2 Building Geometries over Real Algebras

INTRODUCTION

Real Algebras

A commutative algebra over $\mathbb{R}$ is a real vector space A equipped with a bilinear multiplication $\mu\colon A\times A\to A$. An algebra A is commutative if $\mu(a,b)=\mu(b,a)$ for all a,b; A topological if μ is continuous, and of category C if $\mu\in\mathrm{Hom}_{\mathsf{C}}(A^2,A)$. An element $a\in A$ is a left zero divisor if $\mu(a,\cdot)$ has a nontrivial kernel, and a left unit if $\mu(a,\cdot)$ is an isomorphism, analogously for right zero divisors and units. As convention, when not specified we will always mean *left* zero divisor and *left* units. Let $A^\times$ denote the set of units, and A_Z the set of zero divisors. If A is finite dimensional then $A=A^\times\sqcup A_Z$.

Lemma 32: *The zero divisors $A_Z\subset A$ of a topological algebra form a closed subset.*

Proof. Let A be an N-dimensional algebra over $\mathbb{R}$ and $\{z_i\}\subset A_Z$ be a sequence of zero divisors, converging to $z\in A$. For each z_i there is some $w_i\in A\setminus\{0\}$ with $z_iw_i=0$, and in fact all scalar multiples $\mathbb{R}^\times w_i=[w_i]$ satisfy this as well. The sequence $\{[w_i]\}$ subconverges in $(A\smallsetminus\{0\})/\mathbb{R}^\times\cong\mathbb{R}\mathsf{P}^{N-1}$ to $[w]$ by compactness, so choose representatives $w_i\to w$ (say, on the unit sphere). Then as $w_iz_i=0$, continuity of multiplication forces $wz=0$ so $z\in A_Z$. □

Corollary 33: *As every element of a finite dimensional algebra is either a zero divisor or unit, the units $A^\times\subset A$ are an open subset.*

If A is a smooth algebra the group of units $A^\times$ is an open subset, thus a submanifold, and so $A^\times$ is a Lie group. Furthermore any closed subgroup of $A^\times$ is a Lie subgroup. An *involution* on an algebra A is an element $\sigma\in\mathsf{End}(A)$ of order two The action of σ on the underlying vector space satisfies $\sigma^2-1=0$ and decomposes A as a direct sum of the $+1$ and -1 eigenspaces, $A=\mathsf{Fix}(\sigma)\oplus\mathsf{Neg}(\sigma)$. For any choice of $j\in\mathsf{Fix}(\sigma)$ the involution provides a map $\phi_j\colon A\to\mathsf{Fix}(\sigma)$ given by $\phi_j(x)=\sigma(x)jx$. When $j=1$ this map is multiplicative, thus a group homomorphism called the *norm*, $x\mapsto\sigma(x)x$. The preimage of $\{1\}$ is the 1-dimensional *unitary group*, $\mathsf{U}(A):=\{\alpha\in A^\times\mid\sigma(\alpha)\alpha=1\}$.

Given a real algebra A the matrix algebras $\mathsf{M}(n;A)$ are given by imposing matrix multiplication on the spaces A^{n^2}. As this multiplication is built directly out of that of A, the matrix algebras are C-algebras iff A is, respectively. An involution $\sigma\colon A\to A$ extends via component-wise application to $\mathsf{M}(n;A)$ and induces an involution analogous to the conjugate transpose, $X^\dagger=\sigma(X)^T$. The decomposition of $\mathsf{M}(n,A)$ corresponding to $\dagger$ determines the *Hermitian* $\mathsf{Fix}(\dagger)=\mathsf{Herm}(n;A,\sigma)$ and *skew-Hermitian* $\mathsf{Neg}(\dagger)=\mathsf{SkHerm}(n;A,\sigma)$ matrices. For commutative algebras A, the usual formula for the determinant provides a map $\mathsf{det}\colon\mathsf{M}(n,A)\to A$. Cramer's shows $B\in\mathsf{M}(n;A)$ is invertible iff $\mathsf{det}(B)$ is. As det is polynomial in the matrix entries, $\mathsf{det}\in\mathrm{Hom}_{\mathsf{C}}(\mathsf{M}(n,A),A)$ and inversion (given by the matrix of cofactors) is a C-morphism on the complement of $\mathsf{det}^{-1}\{A_Z\}$.

Thus $\mathsf{GL}(n,A)=\mathsf{det}^{-1}\{A^\times\}$, which is an open subset (thus submanifold) of the $\mathsf{M}(n,A)$. The group operations of multiplication and inversion are C-morphisms on

$\mathsf{GL}(n, A)$, providing the structure of a C-group. The determinant provides a group homomorphism $\mathsf{det}\colon \mathsf{GL}(n, A) \to A^\times$ and preimages of subgroups give important subgroups of $\mathsf{GL}(n; A)$. As our interest is particularly in the smooth category, the following provides a method of producing Lie subgroups.

Proposition 34: *Let A be smooth and commutative, then $\mathsf{det}\colon \mathsf{GL}(n; A) \to A^\times$ is a submersion.*

Proof. Let $B \in \mathsf{GL}(n; A)$, then for each $X \in \mathsf{M}(n; A)$ the path $B_t = (I + tX)B$ passes through B and $\frac{\mathsf{d}}{\mathsf{dt}}|_{t=0}\mathsf{det}(B_t) = \mathsf{tr}(X)\mathsf{det}(B)$ so for any $\alpha \in A$ the choice $X_\alpha = \frac{\alpha}{n\mathsf{det}(B)} I$ shows the derivative surjects onto $A = T_{\mathsf{det}B}A^\star$. □

Corollary 35: *If A is a smooth commutative algebra and $G \leqslant A^\times$ a closed subgroup, then $\mathsf{det}^{-1}\{G\}$ is a Lie subgroup of $\mathsf{GL}(n; A)$. In particular the closed subgroup $\{1\} \leqslant A^\times$ corresponds to the special linear group $\mathsf{SL}(n; A) = \mathsf{det}^{-1}\{1\}$.*

Projective Geometry

Classically, projective geometry is given by the projectivization of the linear action of $\mathsf{GL}(n, \mathbb{F})$ on $\mathbb{F}^n$. Taking the group-space viewpoint, this is the action of $\mathsf{GL}(n; \mathbb{F})$ on the projective space $\mathbb{F}\mathsf{P}^{n-1} = (\mathbb{F}^n \smallsetminus 0)/\mathbb{F}^\times$. Taking the automorphism-stabilizer viewpoint, the geometry of projective space corresponds to the pair $(\mathsf{GL}(n, \mathbb{F}), \mathsf{stab})$ with stab the stabilizer of a projective point in $\mathbb{P}(\mathbb{F}^n)$ which realizes projective space as the quotient $\mathbb{P}(\mathbb{F}^n) = \mathsf{GL}(n, \mathbb{F})/\mathsf{stab}$.

The geometry corresponding to $(\mathsf{GL}(n, F), \mathsf{Stab}[p])$ is independent of the choice of point $p \neq 0$ for projective geometry over a field $\mathbb{F}$, but this does not remain true for a general algebra A. We will say that a point $p \in A^n$ is *good* if the point stabilizer is of minimal dimension, and *bad* otherwise. One way to choose good points is as follows. For a point $p \in A^n$ let $I_p \leqslant A$ be the ideal generated by its coordinates, $I_p = \langle p_1, \dots p_n \rangle$. Note that for any $X \in \mathsf{GL}(n; A)$ the ideals I_p and I_{Xp} are identical and so this is an invariant of $\mathsf{GL}(n; A)$ orbits. Conversely if $I_p = I_q$ then each q_i is a A-linear combination of the p_i so $q = Xp$, so in fact the ideal I_p determines the orbit. Generically, $I_p = A$ and strictly smaller ideals appear only when no coordinate (and no linear combination of the coordinates) is a unit. Such points are *bad*, the generic case are the *good points*.

We may also take the group-space perspective, and try to define an analog of projective space over an algebra directly. Here, the analog of $\vec{0} \in \mathbb{F}^n$, are points on which the action of the units $A^\times$ is not free.

Definition 45: *The* generalized zeroes *of A^n are the elements $Z(A^n) \subset A^n$ such that $z \in Z(A^n)$ if for some unit $u \in A^\times$, scalar multiplication by u fixes z.*

As the analogs of zero, we denote this collection by $Z(A^n)$. The points of $A^n \setminus Z(A^n)$ constitute a single $\mathsf{GL}(n, \mathbb{A})$ orbit, and so have isomorphic point stabilizers.

Definition 46: *The projective space AP^n is the quotient of $A^n \smallsetminus Z(A^n)$ by the left action $a.(v_i) = (av_i)$ of $A^\times$.*

Definition 47: *Let A be a finite dimensional commutative algebra over $\mathbb{R}$, and $n \in \mathbb{N}$. Then $\mathsf{St}(n;A)$ is the stabilizer of $(0,\cdots,0,1)$ under the linear action of $\mathsf{GL}(n,A)$ on A^n.*

$$\mathsf{St}(n;A) = \left\{ \begin{pmatrix} X & \vec{0} \\ \vec{v} & \alpha \end{pmatrix} \mid \alpha \in A^\star, v \in A^{n-1}, X \in \mathsf{GL}(n-1;A) \right\}.$$

We denote the intersection $\mathsf{St}(n;A) \cap \mathsf{SL}(n;\mathbb{A}) = \mathsf{SSt}(n;A)$. Note that $(0,\cdots,0,1) \in A^n \setminus Z(A^n)$ for any algebra A, and so we may use $\mathsf{St}(n;A)$ to define projective geometry generally.

Definition 48: *The $(n-1)$ dimensional projective geometry over A is given by the pair $(G,K) = (\mathsf{GL}(n;A), \mathsf{St}(n;A))$, The effective version of this geometry is given by projectivization, $(\mathbb{P}\mathsf{St}(n;A), \mathsf{PGL}(n;A))$ and another convenient incarnation is $(\mathsf{SL}(n;A), \mathsf{SSt}(n;A))$ when A is commutative. The projective space $\mathsf{AP}^{n-1} = \mathbb{P}(A^n)$ is defined as the coset space $\mathsf{GL}(n;A)/\mathsf{St}(n;A)$.*

Alternatively, from the group-space perspective, we have the following equivalent definition.

Definition 49: *The $n-1$ dimensional projective geometry over A has domain $\mathsf{AP}^{n-1} = (A^n \smallsetminus Z(A^n))/\sim$ for $\vec{v} \sim \vec{w}$ if there is an $a \in A^\times$ such that $a\vec{v} = \vec{w}$. The (non-effective) automorphism group is $\mathsf{GL}(n;A)$.*

To see that smooth algebras define *smooth* projective geometries, we need to show that AP^{n-1} is a smooth manifold, or equivalently that $\mathsf{St}(n;A)$ is a Lie subgroup of $\mathsf{GL}(n,A)$. This second fact is immediate from the closed subgroup theorem as $\mathsf{St}(n;A)$ is the intersection of a linear subspace of $\mathsf{M}(n;A)$ with $\mathsf{GL}(n;A)$; however we give an explicit argument which will be used in the generalization to families.

Proposition 36: *The map $\mathsf{GL}(n;A) \to A^{n-1}$ projecting onto the first $n-1$ entries of the last column is a submersion.*

Proof. Let $\pi\colon \mathsf{GL}(n;A) \to A^{n-1}$ be the projection map $(X_{ij}) \mapsto (X_{1,n},\ldots,X_{n-1,n})$. Then for any $B \in \mathsf{GL}(n;A)$ and $v \in A^{n-1} \cong T_{\pi(B)}A^{n-1}$ the path $B_t = B + t\begin{pmatrix} O & \vec{v} \\ \vec{0} & 0 \end{pmatrix}$ has $\frac{\mathsf{d}}{\mathsf{dt}}|_{t=0}\pi(B_t) = v$ so $(D\pi)_B$ is surjective. □

Unitary Geometry

Fix an algebra with involution (A,σ) and a nondegenerate $J \in \mathsf{Herm}(n;A,\sigma)$. A matrix X is said to *preserve* J if $X^\dagger JX = J$. The map $\Phi_J\colon \mathsf{M}(n;A) \to \mathsf{Herm}(n;A,\sigma)$ given by $X \mapsto X^\dagger JX$ defines the *generalized unitary group* for J.

Definition 50: *The generalized unitary group $\mathsf{U}(J,A,\sigma) = \Phi_J^{-1}\{J\}$ consists of the matrices preserving J: $\mathsf{U}(J;A,\sigma) = \{X \mid X^\dagger JX = J\}$.*

The map Φ_J is a C-morphism as it is built out of algebra operations and the involution. Thus in particular $\mathsf{U}(J;A,\sigma)$ is a closed subgroup of $\mathsf{GL}(n;A)$. In the case that A is a smooth algebra, this is enough to conclude the unitary groups are Lie groups. However the following direct argument will prove useful later on.

Lemma 37: *The map* $\Phi_J\colon \mathsf{GL}(n;A) \to \mathsf{Herm}(n;A,\sigma)$ *is a submersion when* A *is a smooth algebra.*

Proof. Let $B \in \mathsf{U}(J;A,\sigma)$, then for any $X \in \mathsf{M}(n,A)$ we may construct the path $B_t = B + tX$ which remains in $\mathsf{GL}(n,A)$ for small t. Computing the derivative we see $\frac{\mathsf{d}}{\mathsf{dt}}|_{t=0}\Phi_J(B_t) = X^\dagger JB + B^\dagger JX$, and so Φ_J is a submersion if $X \mapsto X^\dagger JB + B^\dagger JX$ surjects onto $T_{\Phi_J(B)}\mathsf{Herm}(n;A,\sigma) = \mathsf{Herm}(n;A,\sigma)$. This map is $\mathbb{R}$-linear and so we proceed by dimension count, noting $\dim\operatorname{image}\Phi_J = \dim\mathsf{M}(n,A) - \dim\ker\Phi_J$. The kernel of Φ_J is given by $\ker\Phi_J = \left\{X \mid X^\dagger JB = -B^\dagger JX\right\}$, which as B, J are invertible can be expressed $\ker\Phi_J = (B^\dagger J)^{-1}\mathsf{SkHerm}(n;A,\sigma)$. Thus $\dim\ker\Phi_J$ is the dimension of the space of skew-Hermitian matrices, so the dimension count above shows $\dim\operatorname{image}\Phi_J$ to be the same as the dimension of the space of Hermitian matrices (the complementary subspace to SkHerm in $\mathsf{M}(n,A)$). But $\mathsf{Herm}(n;A,\sigma)$ is the codomain so $(D\Phi_J)_B$ is surjective, and Φ_J is a submersion. □

Taking the determinant of the equation $\Phi_J(X) = J$ gives $\mathsf{det}(X^\dagger)\mathsf{det}(X) = 1$ as J is nondegenerate, and $\mathsf{det}(X^\dagger) = \sigma(\mathsf{det}(X))$ so $\mathsf{det}X \in \mathsf{U}(A,\sigma)$. Thus the determinant restricts to a homomorphism $\mathsf{det}\colon \mathsf{U}(J;A,\sigma) \to \mathsf{U}(A,\sigma)$.

Lemma 38: *The determinant* $\mathsf{det}\colon \mathsf{U}(J;A,\sigma) \to \mathsf{U}(A,\sigma)$ *is a submersion when* A *is a smooth commutative algebra.*

Proof. The determinant is a group homomorphism $\mathsf{U}(J;A) \to \mathsf{U}(A)$ defining the closed subgroup (hence Lie subgroup, and manifold $\mathsf{SU}(J;A)$). Together these three form a short exact sequence

$$1 \to \mathsf{SU}(J,A) \to \mathsf{U}(J;A) \to \mathsf{U}(A) \to 1$$

so topologically $\mathsf{U}(J;A)$ is a product $\mathsf{SU}(J;A) \times \mathsf{U}(A)$ and in these coordinates the determinant is the projection map, which is a smooth submersion. □

Corollary 39: *Preimages of closed subgroups of* $\mathsf{U}(A,\sigma)$ *give Lie subgroups of* $\mathsf{U}(J;A,\sigma)$. *In particular,* $\mathsf{det}|^{-1}_{\mathsf{U}(J;A,\sigma)}\{1\} = \mathsf{SU}(J;A,\sigma)$ *is a Lie subgroup.*

This generalized notion of unitary group encompasses both the classical orthogonal and unitary groups, together with many new examples.

Example 2: Let $A = \mathbb{C}$ and choose the trivial involution $\sigma = \mathsf{id}_{\mathbb{C}}$. Then the unitary groups corresponding to $J = \mathsf{diag}(I_p, -I_q)$ are the classical orthogonal groups, $\mathsf{U}(J,\mathbb{C},\mathsf{id}) = (p,q;\mathbb{C})$. If instead $\sigma(x+iy) = x - iy$ is complex conjugation, the generalized unitary group for J is the classical indefinite unitary group $\mathsf{U}(J;\mathbb{C},\sigma) = \mathsf{U}(p,q;\mathbb{C})$.

The unitary geometries are determined by the action of the groups $\mathsf{U}(J;A,\sigma)$ on AP^n, or equivalently by $\mathsf{U}(J;A,\sigma)$ together with its intersection with a point stabilizer of the $\mathsf{GL}(n+1,\mathbb{A})$ on AP^n.

Definition 51: *A unitary geometry over* (A,σ) *is given by the pair* $(G,C) = (\mathsf{U}(J;A), \mathsf{Stab}([p]) \cap \mathsf{U}(J;A))$ *for* $J \in \mathsf{Herm}(n;A)$ *and* $[p] \in \mathsf{AP}^n$ *and is called the* unitary geometry corresponding to (J,p)

When $p \in \mathsf{AP}^n$ is not on the lightcone of the Hermitian form J (that is, $p^\dagger J p \neq 0$) this embeds as a subgeometry of projective geometry. A priori a unitary geometry depends on both a choice of Hermitian form J and projective point $[p]$, and at times it is useful to be able to vary these two parameters independently. However the choice of point can be absorbed into the choice of Hermitian form as the proposition below shows, which we will often do out of convenience.

Lemma 40: *Let (A, σ) be an algebra with involution, and $J \in \mathsf{Herm}(n; A)$. Then if $p, q \in A^n$ have the unitary geometry corresponding to (J, p) is isomorphic to that of $(C^\dagger JC, q)$ for some $C \in \mathsf{GL}(n; A)$.*

Proof. Let $J \in \mathsf{Herm}(n; A)$ and $p, q \in A^n$. Taking $C \in \mathsf{GL}(n; A)$ with $Cp = q$ note that $\mathsf{stab}(q) = C\mathsf{stab}(p)C^{-1}$ and conjugation by A gives an isomorphism between the group-stabilizer geometries $(\mathsf{U}(J; A), \mathsf{stab}(p))$ and $(C\mathsf{U}(J; A)C^{-1}, C\mathsf{stab}(p)C^{-1})$. But $C\mathsf{U}(J; A)C^{-1} = \mathsf{U}(C^\dagger JC; A)$ and so we have an isomorphism of geometries $(\mathsf{U}(J; A), \mathsf{stab}(p))$ and $(\mathsf{U}(C^\dagger JC; A), \mathsf{stab}(q))$ as claimed.

□

Thus we will fix the point $p = (0, \ldots, 0, 1)$ and talk of *the* unitary geometry corresponding to $\mathsf{U}(J; A)$ as the geometry corresponding to the pair $(J, [p])$.

Definition 52: *The* unitary geometry *for $\mathsf{U}(J; A) \leqslant \mathsf{GL}(n + 1; A)$ is given by the pair $(\mathsf{U}(J; A, \sigma), \mathsf{USt}(J; A))$ for $\mathsf{USt}(J; A) = \mathsf{U}(J; A) \cap \mathsf{St}(n + 1, A)$.*

The fact that $\mathsf{USt}(J; A)$ is a Lie group is obvious as its closed in $\mathsf{St}(n; A)$, but again we give a more detailed argument for future use.

Lemma 41: *The restriction of $\Phi_J : X \mapsto X^\dagger JX$ to $\mathsf{St}(n; A)$ is a submersion onto $\mathsf{Herm}(n; A)$, for J diagonal (surely this restraint can be removed)*

Proof. For clarity write $D\Phi_J = \phi$ and $\mathsf{St}(n; A) = \mathsf{St}$. As St is the intersection of a linear subspace $\overline{\mathsf{St}} \subset \mathsf{M}(n; A)$ with $\mathsf{GL}(n; A)$ for each $B \in \mathsf{St}$ the tangent space $T_B\mathsf{St} = \overline{\mathsf{St}}$. The kernel of the restricted map $\phi_{\overline{\mathsf{St}}}$ is the intersection of $\ker\phi$ with $\overline{\mathsf{St}}$, allowing us to calculate the dimension of the image of using

$$\dim \operatorname{img}(\phi|_{\overline{\mathsf{St}}})_B = \dim \overline{\mathsf{St}} - \dim(\ker(\phi)_B \cap \overline{\mathsf{St}}).$$

Thus calculating the dimension of $\operatorname{img}(\phi|_{\overline{\mathsf{St}}})_B$ amounts to understanding the relationship between $\ker(\phi_B)$ and $\overline{\mathsf{St}}$ in $\mathsf{M}(n; A)$. In particular if these subspaces sum to all of $\mathsf{M}(n; A)$ we are done, as

$$\dim M(n; A) = \dim(\ker\phi + \overline{\mathsf{St}}) = \dim\ker\phi + \dim\overline{\mathsf{St}} - \dim(\ker\phi \cap \overline{\mathsf{St}})$$

$$= \dim\ker\phi + \dim\operatorname{img}(\phi|_{\overline{\mathsf{St}}})$$

By previous work page 35 $\ker\phi$ is the same dimension as the space of skew-Hermitian matrices, which would imply that the image of $\phi|_{\overline{\mathsf{St}}}$ has the same dimension as the Hermitian matrices, which are its codomain so $\phi|_{\overline{\mathsf{St}}}$ is surjective. Thus it only remains to show $\mathsf{M}(n; A) = \ker\phi + \overline{\mathsf{St}}$.

The only restriction on the matrices of $\overline{\mathsf{St}}$ is that the first $n-1$ entries of their last column are zero. Thus it suffices to show that any $v \in A^{n-1}$ can appear as the first $n-1$ entries of the final column of a matrix in $\ker\phi$. Recall from lemma 38 that $\ker\phi = (B^\dagger J)^{-1}\mathsf{SkHerm}(n;A)$, and observe that all but the last entry of the final column of matrices in $\mathsf{SkHerm}(n;A)$ can be arbitrary (the last element must be zero). Then $C = (B^\dagger J)^{-1}$ acts via a homeomorphism $A^n \to A^n$ on vectors, in particular on the last column of matrices in SkHerm.

Specializing now to the case $J \in \mathsf{Diag}$, the matrix $(B^\dagger J)^{-1}$ is of the form $\left(\begin{smallmatrix} X & v \\ 0 & \alpha \end{smallmatrix}\right)$ for $X \in \mathsf{GL}(n-1;\mathbb{R})$, which sends $(\vec{v}, 0)$ to $(Xv + w, 0)$ and restricts to a homeomorphism $A^{n-1} \to A^{n-1}$. Thus any vector can arise as the last column in $\ker\phi$ and we are done. □

Isomorphism Type

In the sections above, we have defined unitary/orthogonal and projective geometries over arbitrary (finite dimensional commutative) real algebras. To begin to tame the maddness we need to develop an understanding of the different flavors of geometry which appear. An algebra A is *decomposable* if is is isomorphic to a nontrivial direct sum of algebras. An algebra with involution (A, σ) is decomposable if $A = A_1 \oplus A_2$ and $\sigma = \sigma_1 \oplus \sigma_2$ decomposes as a direct sum of involutions. The main result of this section is that to understand projective and unitary geometries over algebras, it suffices to understand the indecomposable ones.

Projective Geometries

Proposition 42: *Let $A = A_1 \oplus A_2$ be a direct sum of commutative algebras. Then projective geometry over A decomposes as a direct product of the projective geometries over A_1 and A_1.*

Proof. Let e_1, e_2 be orthogonal primitive idempotents so $A = A_1e_1 + A_2e_2$ as a direct sum. Then $\mathsf{GL}(n, A) = \mathsf{GL}(n, A_1) \oplus \mathsf{GL}(n, A_2)$ and $\mathsf{St}(n;A) = \mathsf{St}(n;A_1) \oplus \mathsf{St}(n;A_2)$ are easily checked, and as the linear action of $\mathsf{St}(n;A)$ on $\mathsf{GL}(n;A)$ by translation preserves this decomposition, $(\mathsf{St}(n;A), \mathsf{GL}(n;A)) \cong (\mathsf{St}(n;A_1), \mathsf{GL}(n;A_1)) \times (\mathsf{St}(n;A_2), \mathsf{GL}(n;A_2))$. □

To understand this decomposition better in terms of spaces it helps to think about the set $Z((A_1 \oplus A_2)^n)$: a point (p_1, p_2) is a 'generalized zero' if $\langle p, q\rangle \neq A_1 \oplus A_2$. This occurs precisely when one of the p_i is in $Z(A_i^n)$, so the complement consists of points (p_1, p_2) with $[p_i] \in \mathsf{A_i P}^{n-1}$. Quotienting by the action of $A^\star = A_1^\star \times A_2^\star$ on this sends (p_1, p_2) to $([p_1], [p_2]) \in \mathsf{A_1P}^{n-1} \times \mathsf{A_2P}^{n-1}$.

Two obvious examples of indecomposable real algebras are $\mathbb{R}$ itself and $\mathbb{C}$, with corresponding projective spaces $\mathbb{R}\mathsf{P}^n$ and $\mathbb{C}\mathsf{P}^n$. The algebra $A = \mathbb{R} \oplus \mathbb{R}$ provides decomposable examples, for instance $(\mathbb{R} \oplus \mathbb{R})P^1$ is a geometry on the torus. A new example is provided by the algebra of dual numbers, $\mathbb{R}_\varepsilon = \mathbb{R}[\varepsilon]/(\varepsilon^2)$ which is an indecomposable two dimensional algebra with nilpotents. Both $\mathbb{R}_\varepsilon\mathsf{P}^n$ and $(\mathbb{R} \oplus \mathbb{R})\mathsf{P}^n$ will be discussed in detail in the final section on applications.

Unitary Geometries

Proposition 43: *If* $A = A_1 \oplus A_2$ *and* σ *preserves the factors* $\sigma_1 \oplus \sigma_2 : A_1 \oplus A_2 \to A_1 \oplus A_2$*, then* $\mathsf{U}(J; A, \sigma) \cong \mathsf{U}(J_1; A_1, \sigma_1) \times \mathsf{U}(J_2; A_2, \sigma_2)$ *decomposes as a product for* $J = J_1 e_1 + J_2 e_2 \in \mathsf{M}(n, A)$.

Proof. First note that $\mathsf{Herm}(n; A, \sigma) = \mathsf{Herm}(n; A_1, \sigma_1) \oplus \mathsf{Herm}(n; A_2, \sigma_2)$ as $J^\dagger = (J_1 e_1 + J_2 e_2)^\dagger = (\sigma_1(J_1)^T e_1 + \sigma_2(J_2)^T e_2)$. Fix a nondegenerate $J = J_1 e_1 + J_2 e_2 \in \mathsf{Herm}(n; A, \sigma)$ and let $X = X_1 e_1 + X_2 e_2 \in \mathsf{U}(J; A, \sigma)$. The condition $X^\dagger J X = J$ decouples as two independent equations along the direct sum decomposition as σ preserves the factors, $X_i^\dagger J_i X_i = J_i$ for $i \in \{1, 2\}$. Thus $X_i \in \mathsf{U}(J_i; A_o, \sigma_i)$ and so the map $X \mapsto (X_1, X_2)$ provides a group homomorphism $\mathsf{U}(J, A, \sigma) \to \mathsf{U}(J_1; A_1, \sigma_1) \times \mathsf{U}(J_2; A_2, \sigma_2)$. By the same reasoning any pair (X_1, X_2) with $X_i \in \mathsf{U}(J_i; A_i, \sigma_i)$ corresponds to an element $X_1 e_1 + X_2 e_2 \in \mathsf{U}(J; A, \sigma)$ so this is an isomorphism. □

As with projective geometries, it suffices to understand the indecompsables. The simplest such case is provided by pairs (A, σ) where A is decomposable but σ does not preserve the decomposition - in particular we are interested in algebras $\Lambda = A \oplus A$ with σ the *swap map* $\sigma(x, y) = (y, x)$. Here rather surprisingly the isomorphism type of the generalized unitary groups $\mathsf{U}(J; A, \sigma)$ is independent of the choice of J.

Proposition 44: *Let* $\Lambda = A \oplus A$ *and* $\sigma : \Lambda \to \Lambda$ *be the coordinate swap map. Then* $\mathsf{U}(J; \Lambda, \sigma) \cong \mathsf{GL}(n, A)$ *for any nondegenerate* σ*-hermitian matrix* J.

Proof. Let $J = J_1 e_1 + J_2 e_2$ be σ-Hermitian, then $(J_1 e_1 + J_2 e_2)^\dagger = J_2^T e_1 + J_1^T e_2$ so $J_1^T = J_2$ and $\mathsf{Herm}(n; \Lambda, \sigma) \cong \mathsf{M}(n, A)$. As $\det(X e_1 + Y e_2) = \det(X) e_1 + \det(Y) e_2$ in $A \oplus A$, the nondegenerate Hermitian matrices arise from $\mathsf{GL}(n, A)$. Given a nondegenerate $\mathsf{J} = J e_1 + J^T e_2 \in \mathsf{Herm}(n, \Lambda, \sigma)$ the corresponding unitary group

$$\mathsf{U}(\mathsf{J}, \Lambda, \sigma) = \left\{ X e_1 + Y e_2 \mid (X e_1 + Y e_2)^\dagger (J e_1 + J^T e_2)(X e_1 + Y e_2) = (J e_1 + J^T e_2) \right\}$$

expanding this component-wise gives the redundant equations $Y^T J X = J$ and $X^T J^T Y = J^T$. Taking the determinant of the first gives $\det(Y)\det(X)\det(J) = \det(J)$ and by the assumption that J is nondegenerate, $\det(Y)\det(X) = 1$ so both X, Y are invertible. Rearranging gives $Y = J^{-T} X^{-T} J^T$ and so all elements of $\mathsf{U}(\mathsf{J}, \Lambda, \sigma)$ are of the form $X e_1 + (J X^{-1} J^{-1})^T e_2$ for some $X \in \mathsf{GL}(n, A)$. Running this argument backwards shows that any $X \in \mathsf{GL}(n; A)$ gives an element $X e_1 + (J X^{-1} J^{-1})^T e_2$ of $\mathsf{U}(\mathsf{J}; \Lambda, \sigma)$ and so $X \mapsto X e_1 + (J X^{-1} J^{-1})^T e_2$ is a bijection $\Phi \colon \mathsf{GL}(n; A) \to \mathsf{U}(\mathsf{J}; \Lambda, \sigma)$. Its an easy check that this is a group homomorphism, and so we're done. □

Corollary 45: *With* Λ, A, σ *as above,* $\mathsf{SU}(J, \Lambda, \sigma) \cong \mathsf{SL}(n, A)$.

Proof. Taking the determinant and simplifying gives $\det(X e_1 + (J X^{-1} J^{-1}) e_2) = \det(X) e_1 + \det(X)^{-1} e_2$. This is only real if $\det(X) = \det(X)^{-1}$, and is only 1 if furthermore $\det(X) = 1$, so the image of $\mathsf{SL}(n; A)$ under Φ is precisely $\mathsf{SU}(J; \Lambda, \sigma)$. □

This result has a natural generalization to involutions of the form $\sigma(x, y) = (\phi(y), \tau(x))$ for ϕ, τ involutions of A. Recall the equalizer of two maps $f, g : X \to X$ is $\mathsf{Eq}(f, g) = \{x \mid f(x) = g(x)\}$.

Proposition 46: *Let $\Lambda = A \oplus A$ and $\sigma\colon \Lambda \to \Lambda$ be of the form $\sigma(x, y) = (\phi(y), \psi(x))$ for ϕ, ψ involutions of A. Then $\mathsf{U}(J; \Lambda, \sigma) \cong \mathsf{Eq}(\Phi, \Psi) \cap \mathsf{GL}(n, A)$ for Φ, Ψ the extensions of ϕ, ψ to $\mathsf{M}(n, \mathbb{A})$ respectively.*

Proof. Proceeding similarly to above, note that $(J_1, J_2) \in \mathsf{Herm}(n, \Lambda, \sigma)$ if $(J_1, J_2)^\dagger = (\phi(J_2)^T, \psi(J_1)^T) = (J_1, J_2)$, so $\phi(J_2)^T = J_1$, $\psi(J_1)^T = J_2$. Applying ψ to the second equation gives $\psi^2(J_1)^T = J_1^T = \psi(J_2)$ and comparing with the transpose of the first gives $J_1^T = \phi(J_2) = \psi(J_2)$ thus $J_2 \in \mathsf{Eq}(\phi, \psi)$ and $\mathsf{Herm}(n; \Lambda, \sigma) = \left\{(\phi(J)^T, J) \mid J \in \mathsf{Eq}(\phi, \psi)\right\}$.

Fix a nondegenerate $\mathsf{J} = (\phi(J)^T, J) \in \mathsf{Herm}(n, \Lambda, \sigma)$ and let $(X, Y) \in \mathsf{U}(\mathsf{J}; \Lambda)$. Then $(X, Y)^\dagger(\phi(J)^T, J)(X, Y) = (\phi(J)^T, J)$ which expands component-wise to the two equations $\Phi(Y)^T\Phi(J)^T X = \Phi(J)^T$ and $\Psi(X)^T JY = J$. Taking the determinant of both equations and using that J is nondegenerate gives that X and Y are invertible, playing around with the equations gives two ways to solve for Y, $J^{-1}\Psi(X)^{-T}J = Y = J^{-1}\Phi(X)^{-T}J$. Thus $\Psi(X) = \Phi(X)$ so $X \in \mathsf{Eq}(\Phi, \Psi)$.

In fact, given any $X \in \mathsf{GL}(n, A) \cap \mathsf{Eq}(\Phi, \Psi)$ the matrix $(X, J^{-1}\Phi(X)^{-T}J)$ is an element of $\mathsf{U}(\mathsf{J}, \Lambda)$ as is easily checked, so the map $f\colon \mathsf{GL}(n; A) \cap \mathsf{Eq}(\Phi, \Psi) \to \mathsf{U}(\mathsf{J}; \Lambda, \sigma)$ is a bijection. That f is a group homomorphism follows immediately from writing down $f(X)f(Y)$ and $f(XY)$.

□

There's a potentially useful perspective to take on this result. The collection $\mathsf{Eq}(\phi, \psi)$ is a subalgebra of A on which $\phi = \psi$ restricts to an involution. We can think of both ϕ and ψ as extensions of this involution to A. In this light, $\mathsf{Eq}(\Phi, \Psi) = \mathsf{M}(n, \mathsf{Eq}(\phi, \psi))$ and $\mathsf{Eq}(\Phi, \Psi) \cap \mathsf{GL}(n; A) = \mathsf{GL}(n, \mathsf{Eq}(\phi, \psi))$. Thus we may more succinctly write the result above as

$$\mathsf{U}(J; \Lambda, \sigma) = \mathsf{GL}(n; \mathsf{Eq}(\phi, \psi))$$

SPECIFIC EXAMPLES

We briefly mention some elementary examples. When $A = \mathbb{R}$ we recover the usual geometries $\mathbb{R}\mathsf{P}^n$ and the pseudo-Riemannian geometries $X(p, q)$ associated to the orthogonal groups $(p, q; \mathbb{R})$ of Chapter ??. When $A = \mathbb{C}$, we recover complex projective geometry $\mathbb{C}\mathsf{P}^n$, the geometry of the complex orthogonal group $(n; \mathbb{C})$ (remember, all orthogonal groups are conjuate over $\mathbb{C}$) and the complex unitary geometries of $\mathsf{U}(p, q; \mathbb{C})$, including complex hyperbolic space.

When $A = \mathbb{R} \oplus \mathbb{R}$, Proposition 42 implies that the associated projective geometries $(\mathbb{R} \oplus \mathbb{R})\mathsf{P}^n \cong \mathbb{R}\mathsf{P}^n \times \mathbb{R}\mathsf{P}^n$ are products of real projective space with itself. Likewise, Proposition 43 to analyze the orthogonal groups, and associated orthogonal geometries over $\mathbb{R} \oplus \mathbb{R}$: they similarly turn out to be products $(p, q; \mathbb{R} \oplus \mathbb{R}) \cong (p, q; \mathbb{R}) \times (p, q; \mathbb{R})$. The unitary geometries over $\mathbb{R} \oplus \mathbb{R}$ with respect to the coordinate swap map are all isomorphic to point-hyperplane projective space, as first noticed in Chapter ??.

As a non-commutative example, we quickly mention the quaternions: as a division ring there are no surprises in defining quaternionic projective geometries, and identically to $\mathbb{C}$ all quaternionic orthogonal groups are conjugate. The generalized unitary groups over the quaternions with respect to quaternionic conjugation are the compact symplectic groups, and in particular $\mathsf{U}(n, 1;)$ is the automorphisms of quaternionic hyperbolic space.

Families of Algebras

Recall that a family of algebras may be thought of as a vector bundle together with a map $\mu : \mathcal{A}\times_\Delta\mathcal{A} \to \mathcal{A}$ restricting slicewise to the multiplication of an algebra structure on $\mathcal{A}_\delta$.

Proposition 47: *The units $\mathcal{A}^\times \to \Delta$ of a family of algebras form a family.*

Proof. We will show $\mathcal{A}^\times \subset \mathcal{A}$ is open, which if $\mathcal{A}_Z$ is the collection of zero divisors of $\mathcal{A}$, is equivalent to showing $\mathcal{A}_Z$ is closed. Let $\{z_i\}$ be a sequence of zero divisors in $\mathcal{A}_Z$ converging to $z \in \mathcal{A}$. Write $\pi(z) = \delta$, and $\pi(z_i) = \delta_i$ for convenience. Forgetting the multiplicative structure $\mathcal{A} \to \Delta$ is a family of real vector spaces, and so by ?? we may choose a compact trivializing neighborhood $\delta \in U$ and $h\colon U \times \mathbb{R}^n \to \mathcal{A}|_U$ a trivialization. The set $\mathcal{A}_Z$ is invariant under real scaling, so we may choose a $w_i \in h(\mathbb{S}^{n-1} \times \{\delta_i\})$ for each z_i such that $z_i w_i = 0$. Thus $\{w_i\} \subset h(\mathbb{S}^{n-1} \times U)$ is a subset of a compact space, subconverging $w_i \to w$. As $w_i z_i = 0$ for all i, $zw = 0$ by continuity of multiplication so z is a zero divisor. ◻

An involution is a map of families $\mathcal{A} \xrightarrow{\sigma} \mathcal{A}$ squaring to the identity and restricting slicewise to an algebra involution. On each algebra $\mathcal{A}_\delta$, the restricted involution σ_δ gives a direct sum decomposition $\mathcal{A}_\delta = \mathsf{Fix}(\sigma_\delta) \oplus \mathsf{Neg}(\sigma_\delta)$. The maps $\Phi_\pm\colon \alpha \mapsto \alpha \pm \sigma(\alpha)$ are the projections onto the factors of this direct sum decomposition.

Proposition 48: *Let $\mathcal{A} \to \Delta$ be a family of algebras with involution $\mathcal{A} \xrightarrow{\sigma} \mathcal{A}$. Then the collections $\mathcal{F}ix(\sigma) = \{\alpha \in \mathcal{A} \mid \alpha = \sigma(\alpha)\}$ and $\mathcal{N}eg(\sigma) = \{\alpha \in \mathcal{A} \mid \sigma(\alpha) = -\alpha\}$ are subfamilies of $\mathcal{A} \to \Delta$.*

Proof. We detail the argument for $\mathcal{F}ix(\sigma)$, the remaining case is argued analogously. We define $\Phi_-(\alpha) = \alpha - \sigma(\alpha)$ on and note that $\mathcal{F}ix(\sigma) = \Phi_-^{-1}\{\mathit{0}(\Delta)\}$ is the preimage of the zero section. Restricted to any fiber, Φ_- is the projection $A \to \mathsf{Neg}(\sigma)$ described previously. Thus when $\mathcal{A} \to \Delta$ is a smooth family of algebras, the restriction of Φ_- to each fiber is a smooth submersion. Applying ??, if a smooth map of families is a submersion fiber-wise, it is itself a submersion, and thus gives $\mathcal{A}$ the structure of a smooth family over $\mathcal{N}eg(\sigma)$. We may then apply observation 1 to pull this family back along the zero section $\mathit{0}\colon \Delta \to \mathcal{N}eg(\sigma) \subset \mathcal{A}$ to get a family $\mathit{0}^\star\mathcal{A} \to \Delta$. The elements of $\mathit{0}^\star\mathcal{A}$ satisfy $\Phi_-(\alpha) = \mathit{0}_{\pi(\alpha)}$ or $\alpha - \sigma(\alpha) = 0$. Thus $\mathit{0}^\star\mathcal{A} = \mathcal{F}ix(\sigma)$. ◻

A family $\mathcal{A} \to \Delta$ gives rise to a family of matrix algebras $\mathcal{M}(n, \mathcal{A}) \to \Delta$, constructed on the underlying space $\mathcal{A}^{n^2} \to \Delta$ by imposing matrix multiplication. An involution σ on $\mathcal{A}$ can be promoted to an involution $\dagger\colon \mathcal{M}(n;\mathcal{A}) \to \mathcal{M}(n;\mathcal{A})$ given by $X^\dagger = \sigma(X)^T$. Applying proposition 48 to $\dagger$ gives the families $\mathcal{Fix}(\dagger) = \mathcal{Herm}(n;\mathcal{A},\sigma)$ and $\mathcal{Neg}(\dagger) = \mathcal{SkHerm}(n;A,\sigma)$ of Hermitian and skew-Hermitian matrices, respectively. The usual formula for the determinant provides a C-map of families $det\colon \mathcal{M}(n,\mathcal{A}) \to \mathcal{A}$.

Two families which we use to illustrate the theory are as follows.

Definition 53: *The family $\Lambda_{\mathbb{R}}$ of 2-dimensional algebras over $\mathbb{R}$ from Chapter ??, $\Lambda_\delta = \mathbb{R}[\lambda]/(\lambda^2 = \delta)$ transitioning from $\mathbb{C}$ when $\delta < 0$ to $\mathbb{R} \oplus \mathbb{R}$ when $\delta > 0$.*

Definition 54: *A* quaternion algebra *over $\mathbb{R}$ is a four dimensional noncommutative real algebra defined by two real parameters $a, b \in \mathbb{R}$. The multiplication on $\mathbb{R}^4 = \mathbb{R}\{1,i,j,k\}$ is defined so that $i^2 = a$ and $j^2 = b$ together with $ij = -ji = k$. When $a = b = -1$ this recovers the usual quaternions.*

Definition 55: *The family $\mathcal{H} \to \mathbb{R}^2$ of quaternion algebras has total space $\mathcal{H} = \mathbb{R}\{1,i,j,k\} \times \mathbb{R}^2$ and multiplication on each $\mathcal{H}_(a,b) = \mathbb{R}\{1,i,j,k\}$ is defined such that $i^2 = a$ and $j^2 = b$. This is a continuous family of algebras transitioning from the usual quaternions when $a, b < 0$ to the algebra of 2×2 matrices when either a or b is > 0.*

7.3 Transitions of Projective Geometries coming from Algebra.

Given a smooth family of algebras, constructing a smooth family of geometries it amounts to showing that the given automorphism and stabilizer groups vary smoothly along with the algebra.

Proposition 49: *Let $\mathcal{A} \to \Delta$ be a smooth family of algebras. Then $\mathcal{GL}(n,\mathcal{A}) \to \Delta$ is a family of Lie groups.*

Proof. The general linear family is the units of the matrix algebra $\mathcal{GL}(n;\mathcal{A}) = \mathcal{M}(n,\mathcal{A})^\star$ and so is an open subset by proposition 47. Thus the restricted projection map gives $\mathcal{GL}(n;\mathcal{A})$ the structure of a smooth family. □

Proposition 50: *Let $\mathcal{A} \to \Delta$ be a smooth family of commutative algebras and $det\colon \mathcal{M}(n,\mathcal{A}) \to \Delta$ the determinant map. Then $\mathcal{GL}(n,\mathcal{A}) \stackrel{det}{\to} \mathcal{A}^\times$ is a family.*

Proof. The determinant is a map of families $\mathcal{M}(n,\mathcal{A}) \to \mathcal{A}$ over Δ, so by ?? it is a submersion if its restriction to the vertical slices are. But this is the content of proposition 34, $\mathsf{GL}(n,A) \to A^\times$ is a submersion for any smooth algebra A. □

Corollary 51: *The groups $\mathcal{SL}(n;\mathcal{A})$ are a subfamily of $\mathcal{GL}(n;\mathcal{A}) \to \Delta$ when $\mathcal{A} \to \Delta$ is commutative.*

Proof. By the previous proposition, $\mathcal{GL}(n;\mathcal{A}) \stackrel{det}{\to} \mathcal{A}^\times$ is a family, and let $\mathcal{1}\colon \Delta \to \mathcal{A}$ be the identity section. Then the pullback $\mathcal{1}^\star\mathcal{GL}(n;\mathcal{A}) \to \Delta$ embeds as the subfamily $\mathcal{SL}(n;\mathcal{A}) \to \Delta$. □

Thus, it remains only to show that the stabilizer subgroups vary smoothly.

Proposition 52: *The stabilizer groups*

$$\mathcal{S}t(n+1;\mathcal{A}) = \left\{ \begin{pmatrix} X & \vec{0} \\ \vec{v} & \alpha \end{pmatrix} \mid X \in \mathcal{GL}(n;\mathcal{A}), \vec{v} \in \mathcal{A}^n; \alpha \in \mathcal{A}^\times \right\}$$

are a subfamily of $\mathcal{GL}(n;\mathcal{A})$.

Proof. The choices of elements $X, \vec{v}$ and α are independent, so topologically $\mathcal{S}t(n+1;\mathcal{A}) = \mathcal{GL}(n;\mathcal{A}) \times_\Delta \mathcal{A}^n \times_\Delta \mathcal{A}^\times$ is a product of families and so is abstractly a family. In line with previous arguments however the map $\mathcal{GL}(n+1;\mathcal{A}) \to \mathcal{A}^n$ sending each matrix to the n first elements of the last column is a submersion as it is one fiberwise page **??** and so the pullback of the zero section $0 \colon \Delta \to \mathcal{A}^n$ is a subfamily of $\mathcal{GL}(n+1;\mathcal{A})$ easily seen to be $\mathcal{S}t(n+1;\mathcal{A})$. □

By similar reasoning, when $\mathcal{A} \to \Delta$ is commutative we can see that the collection $\mathcal{SS}t(n;\mathcal{A}) = \mathcal{S}t(n;\mathcal{A}) \cap \mathcal{SL}(n;\mathcal{A})$ is a subfamily of the special linear family.

Theorem 10: *A smooth family of algebras* $\mathcal{A} \to \Delta$ *determines a smooth family of projective geometries* $\mathcal{A}\mathsf{P}^n \to \Delta$ *for each* $n \in \mathbb{N}$.

This has a lot of instances, one for each family of algebras. In particular, it applies to the $\mathbb{C} \to \mathbb{R} \oplus \mathbb{R}$ transition utilized extensively in Chapters **??** and **??**.

Corollary 53: *The projective spaces* $\Lambda_\delta \mathsf{P}^n$ *form a continuous family of geometries, transitioning from* $\mathbb{C}\mathsf{P}^n$ *to* $(\mathbb{R} \oplus \mathbb{R})\mathsf{P}^n \cong \mathbb{R}\mathsf{P}^n \times \mathbb{R}\mathsf{P}^n$.

In dimension 1, this provides a transition from the geometry of $\mathbb{C}\mathsf{P}^1$ to the torus with an action of $\mathsf{SL}(2;\mathbb{R}) \times \mathsf{SL}(2;\mathbb{R})$. Interpreting these as the boundary of $\mathbb{H}^3$ and AdS^3 respectively, this gives an alternative means of constructing the transition of Danciger [6] in dimension 3.

Corollary 54: *Applying Theorem 10 to the family* $\mathcal{H} \to \mathbb{R}^2$ *of real quaternion algebras gives a transition of quaternionic projective space to projective space defined over* $\mathsf{M}(2;\mathbb{R})$. *It is an interesting future direction to consider what these transitions look like, and in particular analyze* $\mathsf{M}(2;\mathbb{R})\mathsf{P}^n$.

7.4 Transitions of Unitary geometries coming from Algebra

Given a nondegenerate section $\mathcal{J} \colon \Delta \to \mathcal{H}erm(n;\mathcal{A},\sigma)$, one can define for each δ the unitary group $\mathsf{U}(\mathcal{J}_\delta;\mathcal{A}_\delta,\sigma_\delta) \leqslant \mathsf{GL}(n;\mathcal{A}_\delta)$. The union of these is the *generalized unitary family* corresponding to $\mathcal{J}$ over Δ. We check here immediately that this is indeed a family.

Proposition 55: *Let* $(\mathcal{A},\sigma) \to \Delta$ *be a family of algebras and* $\mathcal{J} \colon \Delta \to \mathcal{H}erm(n;\mathcal{A},\sigma)$ *a smooth nondegenerate section. Then* $\mathcal{U}(\mathcal{J};\mathcal{A})$ *is a smooth subfamily of* $\mathcal{GL}(n;\mathcal{A})$.

Proof. The map of families $\Phi_\mathcal{J} \colon \mathcal{GL}(n;\mathcal{A}) \to \mathcal{H}erm(n;\mathcal{A})$ given by $X \mapsto X^\dagger \mathcal{J}_{\pi(X)} X$ is a smooth map, and by lemma 37 is fiber-wise a submersion. Thus by **??** actually gives

$\mathcal{GL}(n;\mathcal{A})$ the structure of a family over $\mathcal{H}erm(n;\mathcal{A})$. The section $\mathcal{J}$ then gives a pull-back family $\mathcal{J}^\star\mathcal{GL}(n;\mathcal{A})$ over Δ, which selects out those matrices in $\mathcal{GL}(n;\mathcal{A})$ such that $\Phi(X) = J_{\pi(X)}$. That is, $X^\dagger\mathcal{J}_{\pi(X)}X = \mathcal{J}_{\pi(X)}$, which is the definition of $\mathcal{U}(\mathcal{J},\mathcal{A})$.

$$\begin{array}{ccc} \mathcal{J}^\star\mathcal{GL}(n;\mathcal{A}) & \hookrightarrow & \mathcal{GL}(n;\mathcal{A}) \\ \downarrow & & \downarrow \Phi_{\mathcal{J}} \\ \Delta & \xhookrightarrow[\mathcal{J}]{} & \mathcal{H}erm(n;\mathcal{A}) \end{array}$$

□

Recalling lemma 38 that for a fixed smooth algebra $\det\colon \mathsf{U}(J;A) \to \mathsf{U}(A)$ is a submersion, applying ?? as above shows the determinant gives $\mathcal{U}(\mathcal{J},\mathcal{A})$ the structure of a family over $\mathcal{U}(\mathcal{A})$. Pulling back along the identity section gives the family of special unitary groups.

Corollary 56: *The special unitary groups $\mathcal{SU}(\mathcal{J};\mathcal{A})$ are a subfamily of $\mathcal{U}(\mathcal{J};\mathcal{A})$.*

Unitary geometries are defined via a pair $(\mathsf{U}(J;A), \mathsf{USt}(J;A))$, and so given a family of algebras $(\mathcal{A},\sigma) \to \Delta$ and a smooth section $\mathcal{J}\colon \Delta \to \mathcal{H}erm^\times(n;\mathcal{A})$ the corresponding collection of geometries is given by $(\mathcal{U}(\mathcal{J};\mathcal{A}), \mathcal{USt}(\mathcal{J};\mathcal{A}))$ for $\mathcal{USt}(\mathcal{J};\mathcal{A}) = \mathcal{St}(n;\mathcal{A}) \cap \mathcal{U}(\mathcal{J};\mathcal{A})$. As we have already studied the unitary families, to see this is a smooth family of geometries it suffices to show that the stabilizers form a subfamily of $\mathcal{U}(\mathcal{J};\mathcal{A})$.

Proposition 57: *The unitary stabilizers $\mathcal{USt}(\mathcal{J};\mathcal{A})$ form a subfamily of $\mathcal{U}(\mathcal{J};\mathcal{A}) \to \Delta$.*

Proof. Let $\Psi\colon \mathcal{U}(\mathcal{J};\mathcal{A}) \to \mathcal{A}^{n-1}$ be the map sending each matrix to the first $n-1$ entries of its last column. This is a map of families over Δ and lemma 41 shows that it is fiberwise a submersion, thus in fact gives $\mathcal{U}(\mathcal{J};\mathcal{A})$ the structure of a family over $\mathcal{A}^{n-1}$. Pulling this family back over the zero section $\mathit{0}\colon \Delta \to \mathcal{A}^{n-1}$ gives the family $\mathit{0}^\star\mathcal{U}(\mathcal{J};\mathcal{A}) \to \Delta$ with total space the intersection $\mathcal{U}(\mathcal{J};\mathcal{A}) \cap \mathcal{St}(n;\mathcal{A})$. □

Theorem 11: *Given a smooth family of algebras $\mathcal{A} \to \Delta$ and a "constant" section $\mathcal{J}\colon \Delta \to \mathcal{H}erm(n;\mathcal{A})$, $\delta \mapsto (J,\delta)$, there is a corresponding smooth family of unitary geometries $(\mathcal{U}(\mathcal{J},\mathcal{A}), \mathcal{UST}(\mathcal{J};\mathcal{A}))$.*

This theorem immediately implies the transition of Chapter ??, among other things.

Corollary 58: *There is a transition $\mathbb{H}^n_{\mathbb{C}}$ to $\mathbb{H}^n_{\mathbb{R}\oplus\mathbb{R}}$ through $\mathbb{H}^n_{\mathbb{R}_\varepsilon}$ by considering the signature $(n,1)$ unitary geometries over $\Lambda_{\mathbb{R}}$.*

But recalling that over $\mathbb{R}\oplus\mathbb{R}$ the signature of a unitary group is not well-defined and all unitary geometries are isomorphic (in fact, they are all isomorphic to point-hyperplane projective space); we also have the following corollary.

Corollary 59: *Given any (p,q); there is a transition from the pseudo-Riemannian unitary geometry of signature (p,q) over $\mathbb{C}$ to Point-Hyperplane projective space.*

Letting the involution in the definition of generalized unitary groups be trivial, we may consider the families of orthogonal geometries along the transition as well. In this case, signature is meaningless over $\mathbb{C}$, and all orthogonal geometries are isomorphic.

Definition 56: *The n dimensional orthogonal geometry over $\mathbb{C}$ is given by the pair* $(\mathsf{SU}(n+1;\mathbb{C}), \mathsf{USt}(n+1;\mathbb{C}))$.

Over $\mathbb{R}\oplus\mathbb{R}$, the trivial involution defining the orthogonal groups implies that they all split as a product: $(p,q;\mathbb{R}\oplus\mathbb{R}) \cong (p,q;\mathbb{R})\times(p,q;\mathbb{R})$, and the corresponding geometry is the product of the pseudo-Riemannian homogeneous geometry of signature (p,q) with itself. Together with the above this gives another class of transitions between homogeneous spaces.

Corollary 60: *For every (p,q) there is a transition between the product geometry of $((p,q), X_{p,q})$ with itself, and the $(p+q-1)$-dimensional complex orthogonal geometry.*

As a specific example, even just thinking on the level of automorphism groups the transition $\mathsf{SO}(2;\Lambda_\delta)$ is interesting.

Example 3: The transition from $(2;\mathbb{C})$ to $(2;\mathbb{R}\oplus\mathbb{R})$ is topologically a transition from two cylinders to four tori, two of them 'coming in from infinity':

We may perform a similar analysis over the family $\mathcal{H}$ of quaternion algebras. Understanding the unitary and orthogonal geometries defined over $\mathsf{M}(2;\mathbb{R})$ is a topic of current research.

Corollary 61: *There is a transition of quaternionic hyperbolic geometry to the signature $(n,1)$ unitary geometry over* $\mathsf{M}(2;\mathbb{R})$.

TODO: Replace some of the arguments above by walking through the step-by-step construction of this in the category of families, following exactly how one would do this in the usual smooth category. The diagrams for this are below.

$$\begin{array}{ccccc}
 & & \mathsf{U}(1;\mathbb{C}) & & \\
 & & \downarrow & & \\
\mathbb{C}^{n+1} & \hookleftarrow & V & \twoheadrightarrow & V/U(1;\mathbb{C}) \\
\downarrow & & \downarrow & & \parallel \\
\mathbb{R} & \longleftarrow & \{1\} & & \mathbb{H}^n_{\mathbb{C}}
\end{array}$$

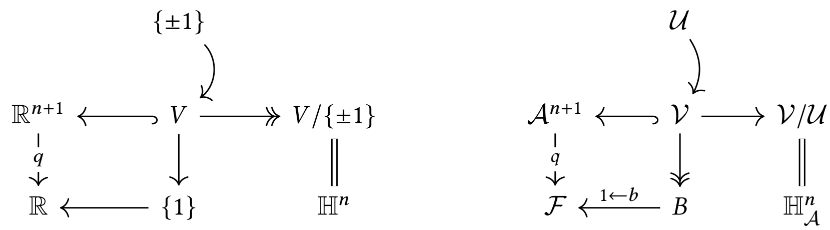

Figure 1: Construction of hyperbolic space and an analog in Fam_B.

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